9 Continuity, Epsilon-Delta Definition

Note 9.1
Thm 3.17 states that if f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

If f is uniformly continuous on $[a, b]$, the graph doesn't wobble too much, and doesnt approach $\pm\infty$ on a bounded interval.
This is because uniform continuity states that if two sequences approach the same value, then the function values at their values must also approach the same value. But if the function increases too quickly, then the distance between the two sequence function values keeps getting larger and larger.

Example 9.2
Take the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} & -1 < x < 1, x \neq 0 \\ 0 & x = 0 \end{cases}$$

We proved that this was continuous:
$x_n \to 0 \implies |f(x_n) - f(0)| = |f(x_n)| = |x_n \sin \frac{1}{x}| \leq |x_n| \to 0$. So f is continuous at 0, so it is continuous on $[-1, 1]$.
Then by the theorem in chapter 3, f is uniformly continuous on $[-1, 1]$.

Example 9.3
Show that $g(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Solution:
Let $u_n = n + \frac{1}{n}$, $v_n = n$ for $n \geq 1$.
Then, $|u_n - v_n| = \frac{1}{n} \to 0$.
But, $|g(u_n) - g(v_n)| = |\left(n + \frac{1}{n}\right)^2 - n^2| = 2 - \frac{1}{n^2} \to 2$.
So, g is not uniformly continuous on $\mathbb{R}$.

Similarly, x^k for $k \geq 2$ is not uniformly continuous on the reals.

Example 9.4
$h(x) = \sqrt{x}$, for $x \geq 0$. h is continuous at 0, and on $[0, \infty)$ h is also uniformly continuous on $[0, \infty)$.

Definition 9.5 ($\epsilon - \delta$ definition of continuity)
$f : D \to \mathbb{R}$ is continuous at x_0 in D if for each $\epsilon > 0$, there is a $\delta > 0$ so that if $|x - x_0| < \delta$, and $x \in D$, then $|f(x) - f(x_0)| < \epsilon$.

Theorem 9.6 (Thm 3.20)
The 2 definitions of continuity are equivalent.

Proof. Assume the $\epsilon - \delta$ definition, and let $u_n \to x_0$, with u_n in domain D of f.
Let $\epsilon > 0$ be arbitrary.
Then by the $\epsilon - \delta$ definition, there is a $\delta > 0$ so if $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \epsilon$.
But, $u_n \to x_0$ implies that there is an N_ϵ so that $n \geq N_\epsilon \implies |u_n - x_0| < \delta$. So, $|f(u_n) - f(x_0)| < \epsilon$, so $f(u_n) \to f(x_0)$.

Next, we will show that the sequence definition of continuity implies the $\epsilon - \delta$ definition.
We will prove this by contradiction for $f : D \to \mathbb{R}$.
There is an x_0 in D and an $\epsilon > 0$ such that for any $n \geq 1$, there is a u_n in D with $|u_n - x_0| < \frac{1}{n}$, but $|f(u_n) - f(x_0)| \geq \epsilon$.
Then $u_n \to x_0$ and $|f(u_n) - f(x_0)|$ and $f(u_n) \not\to f(x_0)$ □

Example 9.7
$f(x) = x^2$. Prove using $\epsilon - \delta$ definition that f is continuous at $x_0 = 3$.

Solution:
Let $\epsilon > 0$ be arbitrary. We want to find $\delta > 0$ so if $(x-3) < \delta$, then $|f(x) - f(3)| < \epsilon$.
Note $|f(x) - f(3)| = |x^2 - 9| = |x-3||x+3|$
Let $\delta = \min(1, \frac{\epsilon}{7})$. Then, $|x-3| < \delta \implies |x-3| < 1 \implies 2 < x < 4 \implies 5 < x+3 < 7$
Then $|x-3| < \delta = \min(1, \epsilon/7) \implies |f(x) - f(3)| = |x-3||x+3| < \frac{\epsilon}{7}(7) = \epsilon$.

Hint for homework:
If we have $g(x) = \sqrt{x}$, $|\sqrt{x} - \sqrt{x_0}| = \left|\frac{x-x_0}{\sqrt{x}+\sqrt{x_0}}\right| \leq \frac{|x-x_0|}{\sqrt{x_0}}$ if $x_0 \neq 0$.

Example 9.8

$$g(x) = \begin{cases} 2x & x \leq 1 \\ 5 - x & x > 1 \end{cases}$$