8 Bisection Method, Uniform Continuity

Example 8.1 (Exercise 3.24)
Let $S \subset R$, $S \neq \varnothing$, S not sequentially compact.

Note that S sequentially compact $\iff$ S is closed and bounded.

So, if S is not sequentially compact, then S is either unbounded, or not closed.

Note that the reals $\mathbb{R}$ is closed.

Every sequence in $\mathbb{N}$ that converges is eventually a repeating sequence, so $\mathbb{N}$ is closed.
Suppose the sequence $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ with $a_n \to n_0$ in $\mathbb{N}$. Eventually, $a_n = n_0$ for all large n.

Note 8.2
Recall that the intermediate value theorem states that for $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then for a c between $f(a), f(b)$, there is an x_0 in (a, b) with $f(x_0) = c$.

We often use the IVT to show that a function has a zero ($f(x_0) = 0$ for some $x_0 \in D$)

Example 8.3
$f(x) = x^3 - 2x^2 + 3x - 7$. Show that f has a zero.

Solution: $f(0) < 0$, $f(1) < 0$, $f(2) < 0$, $f(3) > 0$. f is continuous because it is a polynomial, and so by the IVT, there is an x_0 in $(2, 3)$ with $f(x_0) = 0$.

8.1 Bisection Method

Assume that f is continuous on $[a, b]$, and $f(a) < 0 < f(b)$ (or $f(b) < 0 < f(a)$)

Process: we let $c_1 = \frac{a+b}{2}$, which is the midpoint of $[a, b]$.
Thus, we have 3 cases:
If $f(c_1) = 0$, then we are done.
If $f(c_1) < 0$, then we let $c_2 = \frac{c_1+b}{2}$.
If $f(c_1) > 0$, then we let $c_2 = \frac{a+c_1}{2}$.

And we keep repeating this process for c_n.

We keep cutting our search space in half in each step.

Example 8.4
$f(x) = x^3 - 2x^2 + 3x - 7$. Find $c_3 < \frac{1}{8}$ from zero of f.

Solution: $f(2) < 0$, $f(3) > 0$, so $c_1 = 2.5$.
$f(2.5) > 0$, so we take $c_2 = 2.25$.
$f(2.25) > 0$, so we take $c_3 = 2.125$.

8.2 Uniform Continuity

Definition 8.5
$f : D \to \mathbb{R}$ is **uniformly continuous** on D if for any $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ sequences in D with $|u_n - v_n| \to 0$, then $|f(u_n) - f(v_n)| \to 0$.

Note 8.6
If f is uniformly continuous, then f is continuous.

Proof. Let x_0 be an arbitrary number in the domain of f.
Let $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty} \subseteq D$ and with $|u_n - v_n| \to 0$ and $v_n = x_0$ for all n. a

Then, $|u_n - x_0| \to 0$ and because f is uniformly continuous, we have that $|f(u_n) - f(x_0)| \to 0$, so f is continuous at x_0.

□

Example 8.7
Let $f(x) = x$ for all real x, and let $u_n = n$, $v_n = n + \frac{1}{n}$ for $n \geq 1$.

Then $|u_n - v_n| = |n - (n + \frac{1}{n})| = \frac{1}{n} \to 0$, and $|f(u_n) - f(v_n)| \to 0$.

Theorem 8.8 (Theorem 3.17 **)
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a, b]$.

Proof. By contradiction.
Assume that f is not uniformly continuous, so there is $\epsilon > 0$ and sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty} \subseteq [a, b]$ with $|u_n - v_n| \to 0$, but $|f(u_n) - f(v_n)| \geq \epsilon$ for $n \geq 1$.

Theorem 2.33 implies that there is a subsequence $(u_{n_k})_{k=1}^{\infty}$ converging to some x^* in $[a, b]$.

Also, $|u_n - v_n| \to 0$ implies that $\{v_{n_k}\}_{k=1}^{\infty}$ also converges to x^*.

Because f is continuous on $[a, b]$, we have that $f(u_{n_k}) \to f(x^*)$, and $f(v_{n_k}) \to f(x^*)$. So, $|f(u_{n_k}) - f(v_{n_k})| \to 0$. □

Example 8.9
$h(x) = \frac{1}{x}$, $0 < x < 2$. Then h is not uniformly continuous.

Solution: Let $u_n = \frac{1}{n^2}$, $v_n = \frac{1}{n}$ for $n \geq 1$. Then, $|u_n - v_n| = |\frac{1}{n^2} - \frac{1}{n}| \to 0$.
But, $|h(u_n) - h(v_n)| = |n^2 - n| \to \infty$.

Note 8.10
f being uniformly continuous means that the slope of the graph of f can be "too" steep.

Example 8.11
Let $k(x) = \sin \frac{1}{x}$ for $0 < x < 1$. Then h is not uniformly continuous.

Solution: Let $u_n = \frac{1}{2n\pi}$, $v_n = \frac{1}{2n\pi + \frac{pi}{2}}$. Here, $|u_n - v_n| \to 0|$. But, $|h(u_n) - h(v_n)| = |\sin(2n\pi) - \sin(2n\pi + \frac{pi}{2})| = |0 - 1| = 1$ for all n.

Note 8.12
f being uniformly continuous also means that it can't "wobble" too much.