7 Continuous Functions, Closed Domain Functions

7.1 Continuous Functions

Example 7.1 (Problem 3.1.1d)
True/False: Every function $f : \mathbb{N} \to \mathbb{R}$ is continuous.

Solution: Let $\{x_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$, and let $x_n \to n_0$ (x_n converges to n_0) where n_0 is an arbitrary number in $\mathbb{N}$.
Then there is some $n^* \in \mathbb{N}$ so that $n \geq n^* \implies f(x_n) = f(n_0)$.

Definition 7.2
A function $f : D \to R$ has $f(x_0)$ as a **maximum value** if $x_0 \in D$ and $f(x) \leq f(x_0)$ for all x in D.

A function $f : D \to R$ has $f(z_0)$ as a **minimum value** if $z_0 \in D$ and $f(x) \geq f(z_0)$ for all x in D.

Note 7.3
There can only be 0 or 1 maximum values, and only 0 or 1 minimum values.

Definition 7.4
If $f(x_0) =$ maximum value, then x_0 is a **maximizer**. If $f(z_0) =$ minimum value, then z_0 is a **minimizer**.

Definition 7.5
An **extreme value** of f is either a maximum value or a minimum value.

Example 7.6
$f(x) = \sin x \implies f(\frac{\pi}{2} + 2n\pi) = 1 =$ maximum value, and $f(\frac{3\pi}{2} + 2n\pi) = -1 =$ minimum value for any integer n.
Maximum value of $\sin x$ is 1, minimum value of $\sin x$ is -1.
The maximizers are $\frac{\pi}{2} + 2n\pi$ where n is an integer, and the minimizers are $\frac{3\pi}{2} + 2n\pi$ where n is an integer.

7.2 Closed Domain Functions

Lemma 7.7 (Lemma 3.10)
If $f : [a, b] \to \mathbb{R}$ is continuous, then it is bounded above.

Proof. by contradiction.
Assume f is not bounded above. Then, there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq [a, b]$ with $f(x_n) > n$ for all $n \in \mathbb{N}$.

Since $\{x_n\}_{n=1}^{\infty} \subseteq [a, b]$ is bounded, then by Thm 2.32, there is a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to some x^* in $[a, b]$.

We have that $f(x_{n_k}) \to \infty$, but since f is continuous, we also have that $x_{n_k} \to x^* \implies f(x_{n_k}) \to f(x^*)$. Thus we have a contradiction.

Similarly, $f : [a, b] \to \mathbb{R}$ being continuous implies that f is bounded below, so if f is continuous on a closed interval, then f is bounded. □

Theorem 7.8 (Extreme Value Theorem *** - Thm 3.9)
If $f : [a, b] \to \mathbb{R}$ is continuous, then f has a maximum value and a minimum value.

Proof. By Lemma 3.10, f is bounded. So let $M = \sup\{f(x) : x \in [a,b]\}$.

Then there is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq [a,b]$ with $f(x_n) \to M$. Then $\{x_n\}_{n=1}^{\infty} \subseteq [a,b]$ is bounded, so it has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to some x^* in $[a,b]$.

So, $f(x^*) = \lim_{k\to\infty} f(x_{n_k})$ because f is continuous, and $\lim_{k\to\infty} f(x_{n_k}) = M$. So, f has a maximum value (namely $f(x^*) = M$).

Similarly f has a minimum value. □

Example 7.9
$g(x) = 3\sin(2x)$ for $0 \leq x \leq 2\pi$. Find the maximum value, minimum value, maximizers, and minimizers.

Solution:
Maximum value is $3 = 3\sin(2 \cdot \frac{\pi}{4}) = 3\sin(2 \cdot \frac{5\pi}{4})$. So the maximizers are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

Minimum value is $-3 = 3\sin(2 \cdot \frac{3\pi}{4} 3\sin(2 \cdot \frac{7\pi}{4}$. So the minimizers are $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$.

Example 7.10
Let $h(x) = 2x^3 - 3x^2 - 12x$ for $-2 \leq x \leq 3$.
What is the maximum and minimum value?

We need chapter 4 to answer this question.

Theorem 7.11 (Intermediate Value Theorem *** - Thm 3.11)
Let $f : [a,b] \to \mathbb{R}$ be continuous, and let c be between $f(a)$ and $f(b)$. Then there is an $x_0 \in (a,b)$ with $f(x_0) = c$.

Proof. Assume $f(a) < c < f(b)$, and let $S = \{x \in (a,b) : f(x) < c\}$.
Define $x_0 = \sup S$ (a supremum exists because S is bounded). Note that $a < x_0 < b$.

Suppose that $f(x_0) < c$. Then, there would be an interval $(x_0, x_0 + \delta)$ so that $x \in (x_0, x_0 + \delta)$ implies that $f(x) < c$. But then $x_0 \neq \sup S$, which is a contradiction.

Similarly, suppose that $f(x_0) > c$. Then, there would be an interval $(x_0 - \delta, x_0)$ so that $x \in (x_0 - \delta, x_0)$ implies that $f(x) > c$. So $x_0 \neq \sup S$.

So, $f(x_0) = c$. □

Corollary 7.12
If $f : [a,b] \to \mathbb{R}$ is continuous, then either either (the range of f) $\supseteq [f(a), f(b)]$, or (the range of f) $\supseteq [f(b), f(a)]$, and the range of f is a closed interval.

Corollary 7.13
If f is polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with n odd, then the range of f is $(-\infty, \infty)$.

Proof. If $a_n > 0$, $\lim_{x\to\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$.

Thus we use the intermediate value theorem, where $f(a)$ is very negative and $f(b)$ is very positive, to show that for any real number c, we can find an $a < x_0 < c$ such that $f(x_0) = c$. □