6 Sequences, Continuous Functions

Example 6.1 (Problem 2.3.4)
Assume $a_n \to a$, and $|a| < 1$. Prove $(a_n)^n \to 0$.

Proof. Let $\epsilon > 0$ be arbitrary, with $|a| + \epsilon < 1$.
Then there is an N_ϵ so $n \geq N_\epsilon$ implies that $|a_n| \leq |a_n| + \epsilon$, so $|a_n|^n = |(a_n)^n| \leq (|a_n| + \epsilon)^n \to 0$ by Proposition 2.28. □

Example 6.2 (Problem 2.3.9)
Let $a_n < b_n$ fr all n. Let $I_n = [a_n, b_n]$ with $I_{n+1} \subseteq I_n$ for all $n \geq 1$. Show there are a, b with $a_n \to a$, $b_n \to b$, and $[a, b] \subseteq I_n$ for all n.

Proof. We know that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above by b_n for all $n \geq 1$.
We know that $\{b_n\}_{n=1}^{\infty}$ is decreasing and bounded below by a_n for all $n \geq 1$.

Then the Monotone Convergence Theorem implies that there are a, b with $a_n \to a$, $b_n \to b$. Then $[a, b] \subseteq I_n$ for all n, because if $a_n \leq b_n$ for all n, then $b_n \to b \implies a_n \leq b$ for all n, and $a_n \to a \implies a \leq b_n$ for all n. □

6.1 Continuous Functions - Chapter 3

Definition 6.3
A function $f : D \to R$ is **continuous** at $x_0 \in D$ if whenever $\{x_n\}_{n=1}^{\infty} \subseteq D, x_n \to x_0$, then $f(x_n) \to f(x_0)$

f is a **continuous** function if f is continuous at each x_0 in D.

Example 6.4
Every polynomial is a continuous function (From Polynomial Property in Section 2.1: If p is a polynomial, and a_n converges to a, then $p(a_n) \to p(a)$)

Also, every rational function is a continuous function.

1. The function $f(x) = \frac{1}{x}$ is a continuous function (it is continuous everywhere it is defined, but it is not defined at 0).
2. The function $g(x) = \sqrt{x}$ is a continuous function for the same reasons as above.
3. The function $g(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is a continuous, because if you approach $x = 0$, $g(x)$ approaches 0.
4. The function $h(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous, the function does not approach any significant value as x approaches 0, so it is not continuous at 0 (it keeps oscillating). It is discontinuous even though it is defined for all reals.
5. The function $f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$ is not continuous at 0.

   But, the function $g(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$ is continuous because it is continuous at every nonzero x.

Theorem 6.5 (Continuity Rules)
If f, g are continuous at $x = a$, then $f + g$ and $f \cdot g$ are continuous at $x = a$, and f/g is continuous at $x = a$ if $g(a) \neq 0$.

Example 6.6
Let $f(x) = \sin x$. Show that f is continuous at 0.

Proof. Let x_n converge to 0 ($x_n \to 0$). Then, $0 \leq |\sin x_n| \leq |x_n| \to 0$. So, $\sin x$ is continuous at 0. □

Example 6.7
Let $g(x) = \cos x$. Show that g is continuous at 0.

Proof. $\sin^2 x + \cos^2 x = 1$ for all x.
Let $x_n \to 0$. Then $\cos^2 x_n + \sin^2 x_n \to \cos^2 0 + \sin^2 0 = \cos^2 0$, so $\cos^2 x_n \to 1$.
We have that $\cos x \geq 0$ on $[-\pi/2, \pi/2]$, so we conclude that $\cos x_n \to 1 = \cos 0$. □

Example 6.8
Let $f(x) = \sin x$. Show that f is a continuous function.

Proof. Let x_0 be arbitrary. Let $x_n = x_0 + h_n \to x_0$, so $h_n \to 0$.
Then $\sin x_n = \sin(x_0 + h_n) = \sin x_0 \cancelto{1}{\cos h_n} + \cancelto{0}{\sin h_n} \cos x_0 \to \sin x_0$.
So $\sin x$ is continuous at an arbitrary x_0, so $\sin x$ is a continuous function.

Similarly, $\cos x$ is continuous on $\mathbb{R}$. □

How about $\tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}, \sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x}$? These are also continuous by the earlier continuity rules.

Example 6.9
Let $g(x) = \frac{\sin x}{x}$ if $x \neq 0$, $g(0) = 1$ Prove that g is a continuous function.

Proof. The only problem is at $x = 0$.
Let $x_n \to 0$ (let x_n converge to 0). Show that $g(x_n) \to 1$.

If $x_n > 0$ for all n, then $0 < \sin x_n \leq x_n \implies \frac{\sin x_n}{x_n} \leq 1$.
Also, $x_n \leq \tan x_n = \frac{\sin x_n}{\cos x_n}$. Cross multiply to get $\cos x_n \leq \frac{\sin x_n}{x_n} \leq 1$. But as $n \to \infty$, $\cos x_n \to 1$. So, $x_n \to 0$, $x_n > 0$ for all n implies that $g(0) = \frac{\sin x}{x_n} \to 1$

Similarly, if $x_n < 0$ for some or all $n \geq 1$. So, g is continuous at 0, so g is a continuous function. □

Example 6.10
$k(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.
Then k is not continuous at any real x.