5 Sequences and Subsequences

5.1 Sequences

Proposition 5.1 (Prop 2.28)
Let $|c| < 1$. Show $\lim_{n\to\infty} c^n = 0$

Proof. If $c = 0$, then $c^n = 0$ for all $n \geq 1$.

If $0 < c < 1$, then $\{c^n\}_{n=1}^{\infty}$ is bounded and decreasing.
By the Monotone Convergence Theorem, $c^n \to a \geq 0$.
To show $a = 0$, suppose $a > 0$. Then for each n, $c^n = \frac{c^{n+1}}{c} > \frac{a}{c} > a$, but this means that $\frac{a}{c}$ is a lower bound for c^n which is greater than a, so $a \neq \inf\{c^n\}_{n=1}^{\infty}$. Contradiction.

If $-1 < c < 0$, then look at the above case for $|c|^n \to 0$. □

Theorem 5.2 (Nested Interval Thm ** - Thm 2.29)
Let $I_n = [a_n, b_n]$, with $I_{n+1} \subseteq I_n$ for $n \geq 1$. Assume $\lim_{n\to\infty}(b_n - a_n) = 0$. Then there is a unique x^* so that $\lim_{n\to\infty} a_n = x^* = \lim_{n\to\infty} b_n$.

Proof. $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded (by any b_i), so $\lim_{n\to\infty} a_n = a$ by the monotone convergence theorem. $\{b_n\}_{n=1}^{\infty}$ is decreasing and bounded (by any a_i), so $\lim_{n\to\infty} b_n = b$ by the monotone convergence theorem.

Since $\lim_{n\to\infty}(b_n - a_n) = 0$, then $a = b = x^*$ □

5.2 Subsequences

Definition 5.3
Let $\{a_n\}_{n=1}^{\infty}$ be given, and $\{n_k\}_{k=1}^{\infty}$ a sequence of positive integers, $n_k < n_{k+1}$ for all $k \geq 1$.
If $b_k = a_{n_k}$ for $k \geq 1$, then $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Example 5.4
Let $a_n = \frac{1}{n}$ for $n \geq 1$, and let $b_k = \frac{1}{2^k}$ for $k \geq 1$. Then $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Then $a_n = (-1)^n$ for all $n \geq 1$, which is $\{(-1)^n\}_{n=1}^{\infty}$. Let $\{(-1)^{2n}\}_{n=1}^{\infty}$ be a subsequence. Note that $(-1)^{2n} = 1$.

Definition 5.5
A **peak index** of $\{a_n\}_{n=1}^{\infty}$ is an index m with $a_m \geq a_n$ for all $n \geq m$.

Example 5.6
The peak indices of $\{-1, \frac{1}{2}, -\frac{1}{3}, \cdots\}$ are $2, 4, 6, \cdots$

$\{\frac{n-1}{n}\}_{n=1}^{\infty} : 0, \frac{1}{2}, \frac{2}{3}, \cdots$ has no peak indices since it is an increasing sequence.

Theorem 5.7 (*)
Every sequence $\{a_n\}_{n=1}^{\infty}$ has a monotone subsequence.

Proof. There are a few cases:
Case 1. There are infinitely many peak indices $\{n_k\}_{k=1}^{\infty}$. Then $\{a_{n_k}\}_{k=1}^{\infty}$ is a decreasing subsequence.

Case 2. There are only a finite number of peak indices for $\{a_n\}_{n=1}^{\infty}$. Then there is an n^* so that there are no peak indices for $n \geq n^*$. Thus, there is an increasing subsequence where $n_k \geq n^*$. $\square$

Corollary 5.8 (Thm 2.33)
Every bounded sequence has a convergent subsequence.

Proof. Given $\{a_n\}_{n=1}^{\infty}$ which is bounded, then by the theorem above, there is a monotone subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ So the monotone convergence theorem implies that $\{a_{n_k}\}_{k=1}^{\infty}$ converges. $\square$

Definition 5.9
A set $S \subseteq \mathbb{R}$ is **sequentially compact** if each sequence $\{a_n\}_{n=1}^{\infty} \subset S$ has a subsequence converging to a point in S.

Theorem 5.10 (Thm 2.36)
Each closed and bounded set is sequentially compact.

Proof. Let $\{x_n\}_{n=1}^{\infty} \subseteq S$, with S being closed and bounded. Then $\{x_n\}_{n=1}^{\infty}$ is bounded, and by the preceding theorem, $\{a_n\}_{n=1}^{\infty}$ has a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to some x^*. Since S is closed by hypothesis, and $\{x_{n_k}\}_{k=1}^{\infty} \subseteq S$, then x^* is in S. $\square$

Sequentially compact can be used interchangeably with "closed and bounded," so the professor will no longer be using it.