41 MATH410 Fall 2022 Final Exam

1. (a) Induction: $n^2 \geq 2n+3$ for $n \geq 3$.
   Solution: Base case: $n=3$: $3^2 \geq 2(3)+3$, true.
   Inductive Hypothesis: Assume for an arbitrary $n \geq 3$, $n^2 \geq 2n+3$.
   Then $(n+1)^2 = n^2+2n+1 \geq (2n+3)+2n+1 = 4n+4 \geq_? 2(n+1)+3$.

   (b) Let $\epsilon > 0$. Prove there is an $n > 0$ with $\frac{1}{n} < \epsilon + \sin^2 x$.
   Archimedean property $\implies$ there is n so $0 < \frac{1}{n} < \epsilon$, and thus $0 < \frac{1}{n} < \epsilon + \sin^2 x$.

2. (a) $\{a_n\}_{n=1}^{\infty}$ and a are given.
   Definition: $\lim_{n\to\infty} a_n = a \iff$ for any $\epsilon > 0$ there is an N so $n \geq N \implies |a_n - a| < \epsilon$.
   Negation: There is an $\epsilon > 0$ so for all $N \geq 1$, there is $n_N \geq N$ so $|a_{n_N} - a| \geq \epsilon$.

   (b) $\{b_n\}_{n=1}^{\infty}$ is increasing. $\{b_{n_k}\}_{k=1}^{\infty}$ converges to b.
   Then for any $\epsilon > 0$, there is an N so $n_k \geq N \implies |b_{n_k} - b| < \epsilon$, so $b - \epsilon < b_{n_k} < b + \epsilon$.
   Pick $n_{k_0} \geq N$. Then $n \geq n_{k_0}$ then $\{b_n\}_{n=1}^{\infty}$ increasing implies $b - \epsilon < b_{n_k} \leq b_n < b_n + \epsilon$, so $|b_n - b| < \epsilon$.

3. (a) $g(x) = x^3 + 4x - 7 \implies g(0) < 0, g(2) > 0$. Since g is continuous on $[0,2]$, by the IVT there is a c in $(0,2)$ with $g(c) = 0$.
   $g'(x) = 3x^2 + 4 > 0$ for all x, so g is strictly increasing, so there is only 1 solution.

   (b) f, g is uniformly continuous on D. Let $(u_n)_{n=1}^{\infty}, (v_n)_{n=1}^{\infty}$ be arbitrary in D with $|u_n - v_n| \to 0$.
   Then $|(f+g)(u_n) - (f+g)(v_n)| \leq |f(u_n) - f(v_n)| + |g(u_n) - g(v_n)| \to 0$, since f and g are uniformly continuous.

   (c) $h(x) = \frac{1}{x}$, $x \geq 1$. Let $\epsilon > 0$ be arbitrary. To find $\delta > 0$ so $|x-3| < \delta$, $x \geq 1 \implies \left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon$.
   Note if $\delta = \min(1, \epsilon)$, then $|x - 3| < \delta \implies$

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \frac{|3-x|}{|3x|} \leq \frac{|3-x|}{3} < \frac{\epsilon}{3} < \epsilon$$

4. (a) $f(x) = \frac{1}{\sqrt{x}}$. Want $f'(4)$ by definition.

$$\begin{aligned} f'(4) &= \lim_{x\to 4} \frac{f(x) - f(4)}{x-4} \\ &= \lim_{x\to 4} \frac{\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{4}}}{x-4} \\ &= \lim_{x\to 4} \frac{2-\sqrt{x}}{2\sqrt{x}(x-4)} \\ &= \lim_{x\to 4} \frac{2-\sqrt{x}}{2\sqrt{x}(\sqrt{x}-\sqrt{4})(\sqrt{x}+\sqrt{4})} \\ &= \lim_{x\to 4} \frac{-1}{2\sqrt{x}(\sqrt{x}+2)} \\ &= \frac{-1}{16} \end{aligned}$$

   (b) $g(x) = \sqrt{x}$, $x \geq 1 \implies g'(x) = \frac{1}{2\sqrt{x}} > 0$ for $x > 1$, so g is strictly increasing, so g^{-1} exists.
   $g(4) = \sqrt{4} = 2$. So, $(g^{-1})'(2) = \frac{1}{g'(4)} = 4$.

   (c) MVThm: Let $f : [a,b] \to \mathbb{R}$ be continuous, and differentiable on (a,b). THen there is a c in (a,b) with $f'(c) = \frac{f(b)-f(a)}{b-a}$.
   If $f(x) = \sqrt{x} + x^2$, $1 \leq x \leq 4$, then f satisfies the conditions of MVT, so there is c in $(1,4)$ with

$$f'(c) = \frac{f(4) - f(1)}{4-1} = \frac{16}{3}$$

5. (a) Assume f is bounded on $[a,b]$. Then

$$\underline{\int_a^b} f = \sup_P L(f,P) \leq \inf_Q U(f,Q) = \overline{\int_a^b} f$$

(b) $g(x) = x^3$ for $1 \leq x \leq 2$. Show g is integrable.
Solution: let P_n is a regular partition of $[1, 2]$, with gap$P_n = \frac{1}{n}$. Then $U(g, P_n) - L(f, P_n) = \sum_{i=1}^{n} M_i \frac{1}{n} - \sum_{i=1}^{n} m_i \frac{1}{n} = \frac{1}{n} \sum_{i=1}^{n} (M_1 - m_1 + M_2 - m_2 + \cdots + M_n - m_n) = \frac{1}{n}(M_n - m_1) \to 0$. By the archimedes riemann theorem g is integrable.

(c) $g(x) = \begin{cases} x^2 & 0 \leq x < 2 \\ -1 & x = 2 \end{cases}$ Then g is continuous on $[0, 2)$ and bounded on $[0, 2]$.
Then $\int_0^2 g(x)\, dx = \int_0^2 x^2\, dx = \frac{1}{3}(2)^3 = \frac{8}{3}$.

6. (a) $F(x) = \int_{\ln x}^{\sec x} (3 - t^2)\, dt \implies F'(x) = (3 - sec^2 x)(\sec x \tan x) - (3 - (\ln x)^2) \cdot \frac{1}{x}$.