40 MATH410 Exam 3 Fall 2022

1. (a) $f(x) = \frac{2}{7}x^{7/2}$, $x_0 = 1$. (Note: no derivative at 0 because f is not defined about an interval containing 0). Find $p_2(x)$.

   Solution: $f(x) = \frac{2}{7}x^{7/2}$, $f'(x) = x^{5/2}$, $f''(x) = \frac{5}{2}x^{3/2}$, $f^{(3)}(x) = \frac{15}{4}x^{1/2}$
   $f(1) = \frac{2}{7}$, $f'(1) = 1$, $f''(1) = \frac{5}{2}$. So, $p_2(x) = \frac{2}{7} + (x-1) + \frac{5}{2}\frac{(x-1)^2}{2!}$.

   (b) $f(x) = p_2(x) + r_2(x)$, and $r_2(x) = \frac{f^{(3)}(c_x)}{3!}(x-1)^3$, or $r_2\left(\frac{3}{2}\right) = \frac{f^{(3)}(c_{3/2})}{3!}(\frac{3}{2}-1)^3 = \cdots$

2. (a) $g : \mathbb{R} \to \mathbb{R}$, non constant, $g^{(n)}(0) = 0$ for $n \geq 0$.
   $g(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$.

   (b) Find power series with radius of convergence $= 4$, $I = (-4, 4]$.
   Ex: $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k}\left(\frac{x}{4}\right)^k$.
   By the ratio test, L is $\frac{|x|}{4}\frac{k}{k+1} \to \frac{|x|}{4} < 1$ if $|x| < 4 = r$.
   For I, $x = 4 \implies \sum_{k=0}^{\infty} \frac{(-1)^k}{k}$ converges by the alternating series test.
   $x = -4 \implies \sum_{k=0}^{\infty} \frac{1}{k}$ which diverges by the p test.
   So $I = (-4, 4]$.

3. (a) $\sum_{k=1}^{\infty}(-1)^k\left(\frac{k-1}{k^2+1}\right)^3$ converges absolutely, conditionally, or diveges?

   Solution: $\left|\frac{k-1}{k^2+1}\right|^3 \leq \left(\frac{k}{k^2}\right) = \frac{1}{k^3}$.
   So, $\sum_{k=1}^{\infty}\left|(-1)^k\left(\frac{k-1}{k^2+1}\right)^3\right|| \leq \sum_{k=1}^{\infty}\frac{1}{k^3}$ converges by $p = 3$.
   So the series converges absolutely by the comparison test.

   (b) $f(x) = \frac{1}{1+2x}$. Find power series, r, I, power series for $f'(x)$.

   $\frac{1}{1+x} = \sum_{k=1}^{\infty}(-1)^k x^k \implies \frac{1}{1+2x} = \sum_{k=0}^{\infty}(-1)^k(2x)^k = \sum_{k=0}^{\infty}(-1)^k 2^k x^k$.
   Ratio test: we get $2|x| < 1$ if $|x| < \frac{1}{2}$, so $r = \frac{1}{2}$.
   For I: $x = \frac{1}{2} \implies \sum_{k=0}^{\infty}(-1)^k 2^k(\frac{1}{2})^k = \sum_{k=0}^{\infty}(-1)^k$ diverges.
   $x = -\frac{1}{2} \implies \sum_{k=0}^{\infty} 1$ diverges, so $I = (-\frac{1}{2}, \frac{1}{2})$.
   $f'(x) = \sum_{k=1}^{\infty}(-1)^k 2^k k x^{k-1}$. Note we need to change the bound so we dont get -1.

4. (a) State alternating series test

   (b) Prove if $\{b_n\}_{n=1}^{\infty}$ converges, then it is a cauchy sequence.

   Proof. Let $\epsilon > 0$ be arbitrary. Then $\lim_{n\to\infty} b_n = L \implies$ there is N_ϵ so $n \geq N_\epsilon \implies |b_n - L| < \epsilon$.
   Let $m, n \geq N_\epsilon$. Then $|b_m - b_n| \leq |b_m - L| + |L - b_n| \leq \epsilon + \epsilon = 2\epsilon$. So sequence is Cauchy. □

5. (a) $f_n(x) = 1 - x^n$, $0 \leq x \leq 1$.
   Show $f_n \to f$ pointwise and find the f.
   $0 \leq x < 1 \implies 1 - x^n \to 1$.
   and $f_n(1) = 0$ for all n.
   So if $f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & x = 1 \end{cases}$, then $f_n \to f$ pointwise.

   (b) Note f_n is continuous for all n, and f is not continuous. So $f_n \not\to f$ uniformly. by Thm 9.31.

   (c) $\int_0^1 f_n(x)\,dx = \int_0^1(1-x^n)\,dx = (x - \frac{x^{n+1}}{n+1})|_0^1 = (1 - \frac{1}{1-n+1}) - 0 \to 1 = \int_0^1 f(x)\,dx$. So $\int_0^1 f_n \to \int_0^1 f$.

Example 40.1 (Exercise 9.5.8)
Let $f(x) = \frac{1}{1+x^4}$. Find $\int_0^{1/2} f(x)\,dx$ (power series).

Solution: $\frac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$, $|x| < 1 \implies \frac{1}{1+x^4} = \sum_{k=0}^{\infty}(-1)^k x^{4k}$, $-1 < x < 1$.
$\int_0^{1/2} \frac{1}{1+x^4}\,dx = \int_0^{1/2}\left(\sum_{k=0}^{\infty}(-1)^k x^{4k}\right)\,dx = \sum_{k=0}^{\infty}(-1)^k\left(\int_0^{1/2} x^{4k}\right)\,dx = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+1}}{4k+1}\Big|_0^{1/2} =$
$\sum_{k=0}^{\infty}(-1)^k (1/2)^{4k+1}\frac{1}{4k+1}$.