# 4 Sequences, Closed Sets, Monotonicity

## 4.1 More Sequences

**Note 4.1**
Keep in mind that $a_n$ is not a sequence, it is a number.

**Example 4.2** (Problem 2.1.6)
Suppose $\{a_n\}$ converges to $a > 0$. Show there is an index $N$ so that $n \geq N \implies a_n > 0$.
Solution: Let $\epsilon$ be arbitrary with $\epsilon < \frac{a}{2}$. Then, there is a $N$ such that $n \geq N \implies |a_n - a| < \epsilon < \frac{a}{2}$ so then $a_n > \frac{a}{2} > 0$.

**Example 4.3** (Problem 2.1.14)
Let $s_n = \frac{1}{1(2)} + \frac{1}{2(3)} + \cdots + \frac{1}{n(n+1)}$, for $n \geq 1$. Prove that $\lim_{n\to\infty} s_n = 1$.
$s_n = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n} - \frac{1}{n+1})$. Draw out telescoping cancellations. $= 1 - \frac{1}{n(n+1)} \to 1$

Here, $\{s_n\}_{n=1}^{\infty}$ is a **telescoping sequence**

**Proposition 4.4**
A set $S$ of reals is dense in $\mathbb{R}$ if and only if for each $x$ in $\mathbb{R}$, there is a sequence $\{s_n\}_{n=1}^{\infty} \subseteq S$ with $s_n \to x$.

*Proof.* ($\implies$) Assume $S$ is dense in $\mathbb{R}$, and $x$ is arbitrary in $\mathbb{R}$. For $n \geq 1$, look at $(x - \frac{1}{n}, x + \frac{1}{n})$. Because $S$ is dense in $\mathbb{R}$, this implies that there is an $s_n$ in $(x - \frac{1}{n}, x + \frac{1}{n})$ for $n \geq 1$. Then, $s_n \to x$.

($\impliedby$) Assume $S$ is not dense in $\mathbb{R}$. Then there is an open interval $(a, b)$ such that $(a, b) \cap S = \varnothing$. Then, there is no sequence in $S$ converging to $\frac{a+b}{2}$ $\square$

## 4.2 Closed sets

**Definition 4.5**
A set $S$ in $\mathbb{R}$ is closed if whenever a sequence $\{s_n\}_{n=1}^{\infty} \subseteq S$ and $s_n \to a$, then $a$ is in $S$.

**Example 4.6**
The sets $[1, \pi]$, $\{\frac{1}{n} : n \geq 1\} \cup \{0\}$, and $\{n : n \geq 1\}$ are closed in $\mathbb{R}$.

The sets $(0, 1)$, $[0, 1)$ are not closed in $\mathbb{R}$.

## 4.3 Monotonicity

**Definition 4.7**
$\{a_n\}_{n=1}^{\infty}$ is **increasing** (or monotonically increasing) if $a_n \leq a_{n+1}$ for all $n \geq 1$, and is **decreasing** (or monotonically decreasing) if $a_n \geq a_{n+1}$ for all $n \geq 1$.

$\{a_n\}_{n=1}^{\infty}$ is **monotone** if it is monotonically increasing or monotonically decreasing.

$\{a_n\}_{n=1}^{\infty}$ is **strictly increasing** if $a_n < a_{n+1}$ for $n \geq 1$, and similarly $\{a_n\}_{n=1}^{\infty}$ is **strictly decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.

**Example 4.8**
$\{\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing, $\{1 - \frac{1}{n}\}_{n=1}^{\infty}$ is increasing, $\{\pi, \pi, \cdots\}$ is increasing and decreasing.

**Theorem 4.9** (Monotone Convergence Theorem *** - Thm 2.25)
A monotone sequence converges if and only if it is bounded.

*Proof.* Assume $\{a_n\}_{n=1}^{\infty}$ is increasing.

Assume that $\{a_n\}_{n=1}^{\infty}$ converges.
By Thm 2.18 (Lec 3), then $\{a_n\}_{n=1}^{\infty}$ is bounded.

Assume $\{a_n\}_{n=1}^{\infty}$ is bounded. So let $a = \sup\{a_n : n = 1, 2, \cdots\}$. Let $\epsilon > 0$ be arbitrary. Then there is an $N^*$ so that $a - \epsilon < a_{N^*} \leq a$. But $\{a_n\}_{n=1}^{\infty}$ is increasing by assumption, which implies that if $n \geq N^*$, then $a - \epsilon < a_{N^*} \leq a_n \leq a$, so $|a - a_n| < \epsilon$ if $n \geq N^*$. So, $\lim_{n\to\infty} a_n = a$, so $a_n \to a$. □

**Example 4.10**
Let $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}} = \sqrt{2+a}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} = \sqrt{2+a_2}$, $\cdots, a_{n+1} = \sqrt{2+a_n}$, for $n \geq 1$. Show that $\lim_{n\to\infty} a_n = 2$.

<u>Solution</u>: We use the law of induction to show that $0 \leq a_n \leq 2$ for all $n$, and to show that $\{a_n\}$ is increasing.

We show that $0 \leq a_n \leq 2$ by the law of induction:
Base case: $0 \leq a_1 = \sqrt{2} < 2$.
Induction Hypothesis: assume for arbitrary $n \geq 1$, that $0 \leq a_n \leq 2$.
Then,

$$a_{n+1} = \sqrt{2+a_n} \leq \sqrt{2+2} = \sqrt{4} = 2$$

So, $0 \leq a_n \leq 2$ for $n \geq 1$.

We show that $\{a_n\}_{n=1}^{\infty}$ is increasing by the law of induction:
Base case: $a_1 = \sqrt{2} \leq \sqrt{2+\sqrt{2}} = a_2$.
Induction Hypothesis: assume for arbitrary $n \geq 1$, that $a_n \leq a_{n+1}$.
Then,

$$a_{n+2} = \sqrt{2+a_{n+1}} \geq \sqrt{2+a_n} = a_{n+1}$$

So, $a_{n+2} \geq a_{n+1}$. So, $\{a_n\}$ is increasing.

Then by the Monotone Convergence Theorem, there is an $a$ so that $a_n \to a$, so also, $a_{n+1} \to a$.
So, $a \leftarrow a_{n+1} = \sqrt{2+a_n} \to \sqrt{2+a}$. Thus, $a = \sqrt{2+a}$, so $a^2 = 2+a$, so $a^2 - a - 2 = 0$.
Then, $(a-2)(a+1) = 0$, so $a = 2$ or $a = -1$. We know that all of our $a_n$s are positive, so then we must have $a = 2$, so $\lim_{n\to\infty} a_n = 2$.

**Guiding Question**
Suppose we have $b_1 = \sqrt{6}$, $b_2 = \sqrt{6+b_1}$, $\cdots, b_{n+1} = \sqrt{6+b_n}$. Does $\lim_{n\to\infty} b_n$ exist? If so what is it?