39 Power Series

Suppose $\sum_{k=1}^{\infty} a_k$ converges. What do we know about $\{a_k\}_{k=1}^{\infty}$? $\lim_{k\to\infty} a_k = 0$ and $\{a_k\}_{k=1}^{\infty}$ bounded.

Note 39.1
Recall $\sum_{k=1}^{\infty} a_k$ converges absolutely if $\sum_{k=1}^{\infty} |a_k|$ converges, $\sum_{k=1}^{\infty} a_k$ converges conditionally if $\sum_{k=1}^{\infty} |a_k|$ diverges. (Ex. $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k}$)

Definition 39.2
Let $f(x) = \sum_{k=0}^{\infty} c_k x^k$. The domain of of f is all x such that $\sum_{k=0}^{\infty} c_k x^k$ converges. We say the power series expansion of f is $\sum_{k=0}^{\infty} c_k x^k$ (about $x = 0$).

Theorem 39.3 (Thm 9.40)
If $f(x) = \sum_{k=0}^{\infty} c_k x^k$ converges for $x = s$, then is conveges absolutely for $|x| < |s|$.

Proof. If $f(s) = \sum_{k=0}^{\infty} c_k s^k$ converges, then $\lim_{k\to\infty} c_k s^k = 0$, so there is an $M < \infty$ so $|c_k s^k| \le M$ for $k \ge 0$. Let $|x| < |s|$. Then $\sum_{k=0}^{\infty} |c_k x^k| = \sum_{k=0}^{\infty} |c_k||s^k|\left|\frac{x}{s}\right|^k \le \sum_{k=0}^{\infty} M\left|\frac{x}{s}\right|^k = M\sum_{k=0}^{\infty}\left|\frac{x}{s}\right|^k$ which converges by the geometric series test.

So $\sum_{k=0}^{\infty} c_k x^k$ converges by the comparison test, and converges absolutely. □

Corollary 39.4
Each power series $\sum_{k=0}^{\infty} c_k x^k$ converges in one of the 3 ways:

1. Converges only for $x = 0$. Ex: $\sum_{k=0}^{\infty} x^k$
2. Converges for all x. Ex: $\sum_{k=0}^{\infty} \frac{x^k}{k!}$
3. Converges for all $|x| < r$. Diverges for all $|x| > r$, where $0 < r < \infty$.

Definition 39.5
If $\sum c_k x^k$ converges for $|x| < r$, diverges for $|x| > r$, then r is the **radius of convergence** of the power series.

So, for 1, $r = 0$, for 2, $r = \infty$.

Definition 39.6
The **interval of convergence** of $\sum_{k=0}^{\infty} c_k x^k$, denoted by I, is all x for which the power series converges.

Possibilities: $I = \{0\}$, $I = (-\infty, \infty)$, $I = [-r, r], [-r, r), (-r, r], (-r, r)$, with $0 < r < \infty$.

Example 39.7
$\sum_{k=0}^{\infty} \frac{x^k}{2^k}$
By ratio test: $\left|\frac{x^{k+1}/2^{k+1}}{x^k/2^k}\right| = \frac{|x|}{2} < 1$ if $|x| < 2$, > 1 if $|x| > 2$, so $r = 2$.
For I: $x = 2 \implies \sum_{k=0}^{\infty} \frac{2^k}{2^k} = \sum_{k=0}^{\infty} 1$ diverges by the kth term test.
$x = -2 \implies \sum_{k=0}^{\infty} \frac{(-2)^k}{2^k} = \sum_{k=0}^{\infty} (-1)^k$ diverges by the kth term test.
So, $I = (-2, 2)$.

Example 39.8
$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k3^k}$. By the ratio test:

$$\left|\frac{(-1)^{k+1}x^{k+1}/(k+1)e^{k+1}}{(-1)x^k/x3^k}\right| = \frac{|x|}{3}\frac{k}{k+1} \to \frac{|x|}{3}$$

So for the same reasons as before, $r = 3$.
For I: $x = 3 \implies \sum_{k=1}^{\infty} \frac{(-1)^k 3^k}{k3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ conerges by alternating series test.
$x = -3 \implies \sum \frac{(-1)^k(-3)^k}{k3^k} = \sum \frac{3^k}{x3^k} = \sum \frac{1}{k}$ by kth term test.
$I = (-3, 3]$.

If we have $\sum_{k=0}^{\infty} c_k x^k$, then $\frac{d}{dx}\left(\sum_{k=0}^{\infty} c_k x^k\right) = \sum_{k=1}^{\infty} \frac{d}{dx}(c_k x^k) = \sum_{k=1}^{\infty} kc_k x^{k-1}$. Note here that $k = 1$.

Theorem 39.9 (Thm 9.41)
Let $f(x) = \sum_{k=0}^{\infty} c_k x^k$ with $r > 0$. Then $f'(x) = \sum_{k=1}^{\infty} kc_k x^{k-1}$ has radius r_1, and $r_1 = r$.

Proof. First, show $r_1 \leq r$. Let $0 < |t| < r$. Show $|t| < r_1$.
There is s with $|t| < |s| < r$. Then $\sum_{k=0}^{\infty} c_k s^k$ converges (by Thm 9.40)
Then there is an $M < \infty$ with $|c_k s^k| \leq M$ for all $k \geq 0$.

Then $\sum_{k=1}^{\infty} |c_k k t^{k-1}| = \frac{1}{|s|} \sum_{k=1}^{\infty} |kc_k s^k| \left|\frac{t}{s}\right|^{k-1} \leq \sum_{k=1}^{\infty} kM \left|\frac{t}{s}\right|^{k-1}$.

Ratio test:

$$\left|\frac{(k+1)M(t/s)^k}{kM(t/s)^{k-1}}\right| = \frac{k+1}{k}\left|\frac{t}{s}\right| \to \left|\frac{t}{s}\right| < 1$$

So the series $\sum_{k=1}^{\infty} |c_k k t^{k-1}|$ converges, so $\sum_{k=1}^{\infty} c_k k t^{k-1}$ converges absolutely.

So $|t| < r \implies |t| < r_1$, so $r \leq r_1$.
Proof that $r_1 \leq r$ is in lecture notes, so $r_1 = r$. □

Note 39.10
$I_1 \subseteq I$, and I_1 may not be I.
Example: $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$. Ratio test: $\left|\frac{x^{k+1}/(k+1)}{x^k k}\right| = |x|\frac{k}{k+1} \to |x|$. Then $r = 1$, $I = [-1, 1)$.
$f'(x) = \sum_{k=1}^{\infty} x^{k-1}$, so $r = 1$, $I_1 = (-1, 1)$, so $I_1 \subseteq I$, but $I_1 \neq I$.