38 Uniformly Cauchy

Definition 38.1
If $f_n : D \to \mathbb{R}$ for $n \geq 1$, then the sequence $\{f_n\}_{n=1}^{\infty}$ is **uniformly Cauchy** if for any $\epsilon > 0$, there is N_ϵ so $m, n \geq N_\epsilon \implies |f_m(x) - f_n(x)| < \epsilon$, for all x in D.

Theorem 38.2 (Thm 9.29)
Let $f : D \to \mathbb{R}$ for $n \geq 1$. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly to a function $f : D \to \mathbb{R}$ if and only if $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy.

Proof. ($\implies$) Assume $f_n \to f$ uniformly on D. Then for any $\epsilon > 0$, there is N_ϵ such that if $n \geq N_\epsilon$, then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$, for all x in D.

Then let $m, n \geq N_\epsilon$. Then $|f_m(x) - f_n(x)| \leq |f_m(x) - f(x)| + |f(x) - f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, for all x in D.
Then $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy.

($\impliedby$) Longer proof. We will not prove it here. □

Example 38.3
Let $f_n(x) = \frac{1}{n^{1/2}} \sin(nx)$ for $-1 < x < 1$, and $n \geq 1$.
Then
$$|f_n(x)| = \left| \frac{1}{n^{1/2}} \sin(nx) \right| \leq \frac{1}{n^{1/2} \to 0}$$
Let $f(x) = 0$ for $-1 < x < 1$, then $f_n \to f$ uniformly to f on $(-1, 1)$.
$$|f_n(x) - f(x)| = |f_n(x)| \leq \frac{1}{n^{1/2}} \to 0$$
Note:
$$f_n'(x) = \frac{1}{n^{1/2}} n \cos(nx) = n^{1/2} \cos nx$$
$$|f_n'(x)| = |n^{1/2} \cos nx|$$
for $n \geq 1$, is unbounded for any x in $(-1, 1)$.

Theorem 38.4 (Thm 9.34)
Let I be an open interval, let $f_n : I \to \mathbb{R}$ be continuously differentiable on I. Suppose:

1. $\{f_n\}_{n=1}^{\infty}$ converges pointwise to f on I.
2. $\{f_n'\}_{n=1}^{\infty}$ is uniformly Cauchy on I.

Then $f : I \to \mathbb{R}$ is continuously differentiable, and $\lim_{n\to\infty} f_n'(x) = f'(x)$ for x in I.

Example 38.5 (Example in Section 9.6)
Let f_0 be a tent function on $[0, 1]$: $f_0(0) = 0$, $f_0(\frac{1}{2}) = \frac{1}{2}$, $f_0(1) = 0$
f_0 linear between $0, \frac{1}{2}$ and between $\frac{1}{2}, 1$. $f'(x) = 1$, $0 < x < \frac{1}{2}$, $f'(x) = -1$, $\frac{1}{2} < x < 1$.

Let f_1 have 2 tents on $[0, 1]$.
For $n \geq 1$, let f_n on $[0, 1]$ have 2^n identical tents, one on each subinterval $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$.
$\max f_n(x) = \frac{1}{2^{n+1}}$. $\min f_n(x) = 0$. Note f_n is continuous on $[0, 1]$, for $n \geq 1$.

$s_n = \sum_{k=1}^{n} f_k$ is a continuous function.
Then $\{s_n\}_{n=1}^{\infty}$ converges uniformly to a function f. Then f is continuous on $[0, 1]$. (Note $s_{n+1} - s_n = f_{n+1}$, and $\max f_{n+1} = \frac{1}{2^{n+2}}$)

Example 38.6 (Exercise 9.4.2)
Let $f_n(x) = nxe^{-nx^2}$, $n \geq 1$, $0 \leq x \leq 1$. Show $f_n \to f$ pointwise but $\int_0^1 f_n \not\to \int_0^1 f$.

1st part: Use l'Hopital's rule:

$$\lim_{n\to\infty} nxe^{-nx^2} = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{t\to\infty} \frac{tx}{e^{tx^2}} = \lim_{t\to\infty} \frac{x}{x^2 e^{tx^2}} = 0$$

Let $f(x) = 0$. Then $f_n \to f$ pointwise on $[0, 1]$ since $f_n(0) = 0$.