37 Uniform Convergence of Functions

Example 37.1 (Exercise 9.2.1)

$$f_n(x) = \frac{1-|x|^n}{1+|x|^n} \to \begin{cases} 1 & |x| < 1 \\ 0 & |x| = 1 \\ -1 & |x| > 1 \end{cases} \qquad \text{pointwise}$$

Note that f_n is continuous for all $n \geq 1$, but f is not continuous.

Definition 37.2
Let $f_n : D \to \mathbb{R}$ and $f : D \to \mathbb{R}$.
Then $\{f_n\}_{n=1}^{\infty}$ **converges uniformly** (on D) to f if for any arbitrary $\epsilon > 0$, there is an N_ϵ so that if $n \geq N_\epsilon$, then $|f_n(x) - f(x)| < \epsilon$ for all x in D.

Suppose $f_n \to f$ uniformly on D. Pick any x_0 in D. Then for any x_0 in D, $f_n(x_0) \to f(x_0)$.
Thus $f_n \to f$ uniformly on $D \implies f_n \to f$ pointwise on D.

Theorem 37.3 (Thm 9.31 **)
Let $f_n : D \to \mathbb{R}$ be continuous on D for $n \geq 1$. Let $f_n \to f$ uniformly on D. Then f is continuous.

Proof. Let $\epsilon > 0$ be arbitrary. Since $f_n \to f$ uniformly, there is N_ϵ so $n \geq N_\epsilon \implies |f_n(x) - f(x)| < \frac{\epsilon}{3}$ for x in D.
Pick an arbitrary x_0 in D. We will show f is continuous at x_0.

Note that f_{N_ϵ} is continuous at x_0.
So, there is $\delta > 0$ so that $|x - x_0| < \delta$ and x in $D \implies |f_{N_\epsilon}(x) - f_{N_\epsilon}(x_0)| < \frac{\epsilon}{3}$.

Then, $|f(x) - f(x_0)| \leq |f(x) - f_{N_\epsilon}(x)| + |f_{N_\epsilon}(x) - f_{N_\epsilon}(x_0)| + |f_{N_\epsilon}(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$.

So, f is continuous at x_0, so it is continuous. □

Example 37.4
Let $g_n(x) = x^n$ for $0 \leq x \leq \frac{1}{2}$ and $n \geq 1$. Find f so $f_n \to f$ uniformly on $[0, \frac{1}{2}]$.

Solution: Note: $0 \leq x \leq \frac{1}{2} \implies |g_n(x)| = |x^n| \leq \frac{1}{2^n} \to 0$. Let $g(x) = 0$ for $0 \leq x \leq \frac{1}{2}$.
Then $|g_n(x) - g(x)| = |g_n(x)| \leq \frac{1}{2^n} \to 0$. So $g_n \to g$ uniformly on $[0, \frac{1}{2}]$.

Example 37.5
Let $f_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$. Let $f(x) = e^x$, and let $0 \leq x \leq 1$. Show $f_n \to f$ uniformly on $[0, 1]$.

Solution: Note: $|f(x) - f_n(x)| = |f(x) - p_n(x)| = |R_n(x)| \leq \frac{|f^{(n+1)}(c_x)|}{(n+1)!}|x - 0|^{n+1}$.

$$\frac{|f^{(n+1)}(c_x)|}{(n+1)!}|x-0|^{n+1} \leq \frac{e^{c_x}}{(n+1)!} \leq \frac{e^1}{(n+1)!} \to 0$$

So $|f(x) - f_n(x)| \leq \frac{e}{(n+1)!} \to 0$ for $0 \leq x \leq 1$. So $f_n \to f$ uniformly on $[0, 1]$.

Guiding Question
Suppose $g_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$, $g(x) = e^x$, for all real x. Then does $g_n \to g$ uniformly on $\mathbb{R}$?

Note that here, $|R_n(x)| = \frac{|g^{(n+1)}(c_x)}{(n+1)!}|x|^{n+1} = \frac{e^{c_x}}{(n+1)!}|x|^{n+1} \to \infty$ as $x \to \infty$ and n not too large. So, no.

Theorem 37.6 (Thm 9.32)
Let $a < b$, f_n integrable on $[a, b]$, $f_n \to f$ uniformly on $[a, b]$. Then f is integrable and $\int_a^b f_n \to \int_a^b f$.

Proof. Let $\epsilon > 0$. Then $f_n \to f$ uniformly $\implies$ there is an N_ϵ so $|f_{N_\epsilon}(x) - f(x)| < \frac{\epsilon}{3(b-a)}$.

Since $\int_a^b f_{N_\epsilon}$ exists, there is a $P = \{a = x_0, \cdots, x_n = b\}$ with

$$|U(f_{N_\epsilon}, P) - L(f_{N_\epsilon}, P)| < \frac{\epsilon}{3}$$

Also,

$$|U(f, P) - U(f_{N_\epsilon}, P)| < \frac{\epsilon}{3}$$

and

$$|L(f, P) - L(f_{N_\epsilon}, P)| < \frac{\epsilon}{3}$$

Then,

$$|U(f, P) - L(f, P)| \leq |U(f, P) - U(f_{N_\epsilon}, P)| + |U(f_{N_\epsilon}, P) - L(f_{N_\epsilon}, P)| + |L(f, P) - L(f_{N_\epsilon}, P)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$$

Thus f is integrable on $[a, b]$ by Archimedes Riemann Theorem, and $\int_a^b f_n \to \int_a^b f$. $\square$