36 Convergence Tests (Root Test), Pointwise Convergence

Alternating series test: Let $\{a_k\}_{k=1}^{\infty}$ be monotone decreasing, $\lim_{k\to\infty} a_k = 0$. Then $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.

Also,

$$\left|\sum_{k=1}^{\infty}(-1)^k a_k - \sum_{k=1}^{j}(-1)^k a_k\right| < a_{j+1}$$

For any $j > 1$, where a_{j+1} is the **truncation error**.

Proof.

$$\left|\sum_{k=1}^{\infty}(-1)^k a_k - \sum_{k=1}^{j}(-1)^k a_k\right| = |(a_{j+1} - a_{j+2}) + (a_{j+3} - a_{j+4}) + \cdots|$$

All $(a_{j+1} - a_{j+2}), \cdots$ are positive, so

$$|a_{j+1} - a_{j+2} + a_{j+3} - a_{j+4} + \cdots| = a_{j+1} + (-a_{j+2} + a_{j+3}) + \cdots < a_{j+1}$$

Because all $-a_{j+2} + a_{j+3}, \cdots$ are negative. □

Example 36.1
Given $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k^2}$. Find the smallest reasonable j so that $\left|\sum_{k=1}^{\infty}(-1)^k a_k - \sum_{k=1}^{j}(-1)^k a_k\right| < \frac{1}{100}$.

Solution: to find j so $a_{j+1} < \frac{1}{100}$. This is equivalent to $100 < (j+1)^2$, or $10 < j+1$, , so $9 < j$, so let $j = 10$.

Root Test: Consider $\sum_{k=1}^{\infty} a_k$ with $a_k \geq 0$ for all k. If $\lim_{k\to\infty}(a_k)^{1/k} = L$, then series converges if $L < 1$, diverges if $L > 1$, and inconclusive if $L = 1$.

Proof. if $L < 1$. Let $0 \leq L < 1$, and let $L < a < 1$. Then $\lim_{k\to\infty}(a_k)^{1/k} = L \implies$ there is an N so $k \geq N \implies (a_k)^{1/k} < a$.

Then $a_k < a^k$ for $k \geq N$, so $\sum_{k=N}^{\infty} a_k \leq \sum_{k=N}^{\infty} a^k$. $\sum_{k=N}^{\infty} a^k$ converges $(a < 1)$ by the geometric series test. So $\sum_{k=N}^{\infty} a_k$ converges by the comparison test.

So $\sum_{k=1}^{\infty} a_k$ converges. □

Example 36.2
Show $\sum_{k=1}^{\infty} k\frac{2^k}{3^k}$ converges.

Solution: Root test:

$$\left(k\frac{2^k}{3^k}\right)^{1/k} = k^{1/k} \cdot \frac{2}{3} \to \frac{2}{3}$$

So series converges by the root test.

Example 36.3 (Exercise 8.1.8)
Suppose $\sum_{k=1}^{\infty} a_k$, $\sum_{k=1}^{\infty} b_k$ are positive series, and $\lim_{k\to\infty} \frac{a_k}{b_k} = L$ with $0 < L < \infty$.
Then $\sum_{k=1}^{\infty} a_k$ converges iff $\sum_{k=1}^{\infty} b_k$ converges.

Proof. Let $\epsilon > 0$ be arbitrary with $L - \epsilon > 0$. Then there is an N so that $(L-\epsilon)b_k < a_k < (L+\epsilon)b_k$.

So

$$(L-\epsilon)\sum_{k=N}^{\infty} b_k < \sum_{k=N}^{\infty} a_k < (L+\epsilon)\sum_{k=N}^{\infty} b_k$$

So $\sum_{k=1}^{\infty} b_k$ converges iff $\sum_{k=1}^{\infty} a_k$. □

Example 36.4
Consider $\sum_{k=1}^{\infty} \frac{1}{k}$ and $\sum_{k=1}^{\infty} \sin \frac{1}{k}$. Then

$$\lim_{k\to\infty} \frac{\sin \frac{1}{k}}{1/k} = \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, then by Ex 8.1.8, $\sum_{k=1}^{\infty} \sin \frac{1}{k}$ also diverges.

Definition 36.5
If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} a_k$ **converges absolutely** if $\sum_{k=1}^{\infty} |a_k|$ converges.

And $\sum_{k=1}^{\infty}$ **converges conditionally** if $\sum_{k=1}^{\infty} |a_k|$ diverges.

Note 36.6
$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges, and $\sum_{k=1}^{\infty} \left|(-1)^k \frac{1}{k}\right| = \sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges conditionally.

$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ converges, and $\sum_{k=1}^{\infty} \left|(-1)^k \frac{1}{k^2}\right| = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so $\sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ converges absolutely.

36.1 Section 9.2

Definition 36.7
Let $f_n : D \to \mathbb{R}$ for $n \geq 1$, and $f : D \to \mathbb{R}$. Then f_n converges **pointwise** to f ($f_n \to f$ pointwise) if $f_n(x) \to f(x)$ for all $x \in D$.

Example 36.8
$f_n(x) = x^n$ for $0 \leq x \leq 1$. Find f so $f_n \to f$ pointwise on $[0, 1]$.

Solution: $\lim_{n\to\infty} f_n(x) = 0$ if $0 \leq x < 1$, and $\lim_{n\to\infty} f_n(1) = 1$. So $f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$.
Then $f_n \to f$ pointwise. Note: f_n is continuous for all n, but f is not continuous.

Example 36.9
Let $f_n(0) = 0$, $f_n(1/n) = n$, $f_n(x) = 0$ for $\frac{2}{n} \leq x \leq 1$, and f_n linear on $[0, \frac{1}{n}]$ and on $[\frac{1}{n}, \frac{2}{n}]$.

Then

$$\int_0^1 f_n(x)\, dx = \frac{1}{2}(n)\frac{2}{n} = 1$$

for all n.
Let $f(x) = 0$ for $0 \leq x \leq 1$. Then $f_n \to f$ pointwise, but $\int_0^1 f_n \not\to \int_0^1 f = 0$.

Note 36.10
If a function is continuous except at a finite number of points, then it is integrable.