# 35 Convergence Tests

**Lemma 35.1** (Lemma 8.20)
Assume $c_n > 0$ for $n \geq 1$. If $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = L$ where $L = 1$, then what is $\lim_{n\to\infty} c_n$?

Ex 1: let $a$ be any number $> 0$. Then let $c_n = a$ for all $n$. Then $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1$ and $\lim_{n\to\infty} c_n = a$.
Ex 2: Let $c_n = n$ for $n \geq 1$.Then $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1$ and $\lim_{n\to\infty} c_n = \infty$.
Also $c_n = \frac{1}{n}, \cdots, \lim_{n\to\infty} c_n = 0$.

## 35.1 7 Convergence Tests

1. $k$th term test: If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges.

**Example 35.2**
$\sum_{k=1}^{\infty} k \sin \frac{1}{k}$. Suppose we write

$$\lim_{k\to\infty} k \sin \frac{1}{k} = \lim_{k\to\infty} \frac{\sin \frac{1}{k}}{1/k} = \lim_{m\to\infty} \frac{\sin m}{m} \cdots$$

Note L'Hopital's rule does not apply to sequences.

Valid:
$$\lim_{k\to\infty} k \sin \frac{1}{k} = \lim_{k\to\infty} k \sin \frac{1}{k} = \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\cos x}{1} = 1$$

So diverges by $k$th term test.

2. Comparison test: If $0 < a_k \leq b_k$ for $k \geq 1$, then
   (a) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
   (b) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

**Example 35.3**
$\sum_{k=1}^{\infty} \frac{1}{2^k+1} \leq \sum_{k=1}^{\infty} \frac{1}{2_k}$ which converges, so the given series converges by the comparison test.

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverging means $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ also diverges, because $\frac{1}{k} \leq \frac{1}{\sqrt{k}}$.

3. Geometric Series Test: If $c \neq 0$, then $\sum_{k=m}^{\infty} cr^k = \frac{cr^m}{1-r} = \frac{\text{first term}}{1-r}$ converges if $|r| < 1$. Diverges if $|r| \geq 1$.

**Example 35.4**

$$\sum_{k=2}^{\infty} \frac{2^k + 4^{k+1}}{6^k} = \sum_{k=2}^{\infty} \frac{2^k}{6^k} + \sum_{k=2}^{\infty} \frac{4^{k+1}}{6^k} = \sum_{k=2}^{\infty} \left(\frac{1}{3}\right)^k + 4\sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^k = \frac{1/9}{1-\frac{1}{3}} + 4\frac{4/9}{1-\frac{2}{3}} = \cdots = \frac{11}{2}$$

4. Ratio Test: Let $a_k > 0$, $k \geq 1$, let $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L$
   (a) If $L < 1$, the series converges.
   (b) If $L > 1$, then the series diverges.
   (c) If $L = 1$, then the test is inconclusive.

   Partial proof: (i) Let $0 \leq L < 1$. Then let $a$ be such that $L < a < 1$.
   Then there is $N$ so $k \geq N \implies 0 < \frac{a_{k+1}}{a_k} < a$, so $a_{k+1} < a_k a$. Then $a_{N+1} < a_N a$, $a_{N+2} < a_{N+1} a < a_N a^2$.
   And if $k \geq 0$, then $a_{N+k} \leq a_N a^k$. Then $\sum_{k=N}^{\infty} a_k \leq \sum_{k=N}^{\infty} a_n a^K$ by converges by the geometric series test.
   So by the comparison test, because $\sum_{k=N}^{\infty} a_k$ converges, so $\sum_{k=1}^{\infty} a_k$ converges.

**Example 35.5**
Prove $\sum_{k=1}^{\infty} \frac{k!}{k^k}$ converges.

Proof: by ratio test,

$$\frac{(k+1)!/(k+1)^{k+1}}{k!/k^k} = \frac{(k+1)!}{k!} \frac{k^k}{(k+1)^{k+1}} = \left(\frac{k}{k+1}\right)^k = \left(\frac{1}{1+1/k}\right)^k = \frac{1}{(1+1/k)^k} \to \frac{1}{e} < 1$$

so the series converges.

5. Let $\{a_k\}_{k=1}^{\infty}$ be strictly decreasing, with $\lim_{k=1} a_k = 0$. Let $f : [1, \infty) \to \mathbb{R}$ be continuous , decreasing, $f(k) = a_k$ for $k \geq 1$. Then $\sum_{k=1}^{\infty} a_k$ converges iff $\int_1^{\infty} f(x)\, dx$ converges.

   *Proof.* Claim $\sum_{k=2}^{n} a_k \leq \int_1^n f(x)\, dx \leq \sum_{k=1}^{n-1} a_k$.
   $\sum_{k=2}^{n} a_k$ is the left sum on $[1, n]$ with the partition $\{1, 2, \cdots, n\}$, and $\sum_{k=1}^{n-1} a_k$ is the right sum. Because $f$ is decreasing, the left sum is greater than the integral, and the right sum is less than the integral.
   All 3 of these converge, or none of them converge. □

6. p-test: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges iff $p > 1$. (Proof by integral test with $\int_1^{\infty} \frac{1}{k^p}\, dx$).

**Example 35.6**
$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges ($p = 1$), $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges ($p = 2$.)

7. Alternating Series Test: Let $\{a_k\}_{k=1}^{\infty}$ have $a_{k+1} > a_k$ for all $k \geq 1$, and $\lim_{k\to\infty} a_k = 0$.
   Then $\sum_{k=1}^{\infty} (-1)^k a_k$ converges and $\left| \sum_{k=1}^{\infty} (-1)^k a_k - \sum_{k=1}^{j} (-1)^k a_k \right| < a_{j+1}$ for any $j \geq 2$. (where the second term is the $j$th partial sum.)