34 Name of Lecture

Lemma 34.1 (Lemma 8.20)
Let $\{c_n\}_{n=1}^{\infty}$ be a sequence, with $c_n > 0$ for all n. Assume $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = L$. Then

1. If $0 \le L < 1$, then $\lim_{n\to\infty} c_n = 0$
2. If $L > 1$, then $\lim_{n\to\infty} c_n = \infty$

Proof. If $0 \le L < 1$, then let $L < a < 1$.
Then there is an N so that $n \ge N \implies \frac{c_{n+1}}{c_n} < a$, so $c_{n+1} < c_n a$.
Then $c_{n+2} < c_{n+1}a < c_n a^2$, and for $k \ge 1$, $c_{n+k} < c_n a^k \to 0$. Thus, $\lim_{n\to\infty} c_n = 0$.

If $L > 1$, then let $L > a > 1$.
Then there is an N so that $n \ge N \implies \frac{c_{n+1}}{c_n} > a$, so $c_{n+1} > c_n a$.
Then $c_{n+k} > c_n a^k \to \infty$. □

Question (Homework). What are the possibilities for $\lim_{n\to\infty} c_n$ if $L = 1$?

34.1 Section 9.1

A sequence is a function on all integers $n \ge n_0$ (usually $n_0 = 0$ or $n_0 = 1$).
A sequence $\{a_n\}_{n=1}^{\infty}$ **converges** to a if for any arbitrary $\epsilon > 0$, there is N_ϵ so $n \ge N_\epsilon \implies |a_n - a| < \epsilon$.

Definition 34.2
A sequence $\{a_n\}_{n=1}^{\infty}$ is a **Cauchy sequence** if for each $\epsilon > 0$ there is N_ϵ so $m, n \ge N_\epsilon \implies |a_m - a_n| < \epsilon$.

Note 34.3
$\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence, $\{n + \frac{1}{n}\}_{n=1}^{\infty}$ is not Cauchy.

Proposition 34.4 (Prop 9.2)
Every convergent sequence is a Cauchy sequence.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ converge to L, and let $\epsilon > 0$ be arbitrary.
Then there is an N_ϵ so $n \ge N_\epsilon \implies |a_n - L| < \frac{\epsilon}{2}$.

Then $m, n \ge N_\epsilon \implies |a_m - a_n| \le |a_m - L| + |L - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
So, $\{a_n\}_{n=1}^{\infty}$ is Cauchy. □

Lemma 34.5 (Lemma 9.3)
Each Cauchy sequence is bounded.

Proof. Let $\epsilon > 0$ be arbitrary, and $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence.
Then there is an N so that if $m, n \ge N$, then $|a_m - a_n| < \epsilon$, so $|a_m| < |a_n| + \epsilon$, or $|a_n| \le |a_N| + \epsilon$ if $n \ge N$.

Let $M = \sup\{|a_1|, |a_2|, \cdots, |a_N|, |a_N| + \epsilon\}$. Since M is a number, then $\{a_n\}_{n=1}^{\infty}$ is bounded. □

Theorem 34.6 (Thm 9.4)
A sequence $\{a_n\}_{n=1}^{\infty}$ is convergent if and only if it is Cauchy.

Proof. $\implies$ proved by 9.2
$\impliedby$ Assume $\{a_n\}_{n=1}^{\infty}$ is Cauchy, and let $\epsilon > 0$ be arbitrary.

Then there is an N^* so $m, n \geq N^* \implies |a_m - a_n| < \frac{\epsilon}{2}$.

Since $\{a_n\}_{n=1}^{\infty}$ is bounded by Lemma 9.3, then by the Sequential Comapctness Theorem (2.36), there is a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converging to some L. If $N \geq N^*$, if $n_k \geq N$, then $|a_{n_k} - L| < \epsilon$.
If $n \geq N \geq N^*$, then $|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$.

So $\{a_n\}_{n=1}^{\infty}$ converges to L. □

Definition 34.7
Let $\{a_k\}_{k=1}^{\infty}$ be a sequence, and $s_n = \sum_{k=1}^{n} a_k = n$th partial sum of series $\sum_{k=1}^{\infty} a_k$.

If $\lim_{n\to\infty} s_n = L$, then $\lim_{n\to\infty} s_n = \sum_{k=1}^{\infty} a_k$, so $\sum_{k=1}^{\infty}$ **converges**. Otherwise, $\sum_{k=1}^{\infty} a_k$ **diverges**.

Example 34.8

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Since

$$\sum_{k=1}^{n} \frac{1}{2^k} = \frac{2^n - 1}{2^n} \to 1$$

Example 34.9

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&\geq 1 + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= \infty
\end{aligned}$$

34.1.1 Convergence Tests

1. kth Term Test: If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ automatically diverges.

   Proof. Consider $\sum_{k=1}^{\infty} a_k$ with $\lim_{k\to\infty} a_k \neq 0$.
   Then there is $\epsilon > 0$ so $s_n - s_{n-1} = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = a_n >$ some $\epsilon > 0$ for infinitely many n.
   So $\sum_{k=1}^{\infty} a_k$ diverges. □

2. Comparison Test: Let $0 \leq a_k \leq b_k$ for $k \geq 1$. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

   Proof. Note that $s_n = \sum_{k=1}^{n} a_k \leq \sum_{k=1}^{\infty} b_k$, so $\{s_n\}_{n=1}^{\infty}$ is bounded, and $\{s_n\}_{n=1}^{\infty}$ is increasing because $a_k \geq 0$.

   Then the monotone convergence theorem implies that $\{s_n\}_{n=1}^{\infty}$ converges. □

3. Comparison Test (ii): If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.