33 Name of Lecture

Theorem 33.1
Let $f : I \to \mathbb{R}$, I open interval, with x_0 in I. Assume $f^{(n)}(x_0)$ exists, for all $n \geq 0$.

If there are $r > 0$, $M > 0$ with $|f^{(n)}(x)| \leq M^n$ for all x in I, with $|x - x_0| < r$, and $n \geq 0$, then

$$\lim_{n\to\infty} R_n(x) = 0$$

for all x in I, so

$$f(x) = \lim_{n\to\infty} p_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Example 33.2
$f(x) = e^x$, $x_0 = 0$ implies

$$\lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-0)^{n+1} = \lim_{n\to\infty} \frac{e^{c_x}}{(n+1)!}x^{n+1} = 0 \quad \text{for all } x$$

Suppose $|g^{(n)}(x)| \leq M^n L$ for $n \geq 0$ if $|x - x_0| < r|$. Then

$$|R_n(x)| = \frac{|g^{(n+1)}(c_x)|}{(n+1)!}|x - x_0|^{n+1} \leq \left|\frac{M^{n+1}L}{(n+1)!}\right| |x - x_0|^{n+1}$$

Example 33.3
Let $g(x) = e^{2x}$. Show $\lim_{n\to\infty} R_n(x) = 0$ for all x.

Solution: $g'(x) = 2e^{2x}, \cdots$, and $|g^{(n+1)}(x)| = 2^{n+1}e^{2x}$, so $|R_n(x)| \leq \left|\frac{2^{n+1}e^{2c_x}}{(n+1)!}\right| |x|^{n+1} \to 0$

Let $f(x) = \ln(1 + x)$. To find the power series for $\ln(1 + x)$ for $-1 < x < 1$.
Let $0 < r < 1$. Then $\frac{1}{1-r} = 1 + r + \cdots + r^n + \frac{r^{n+1}}{1-r} = \sum_{k=0}^{n} x^k + \frac{r^{n+1}}{1-r}$.

If $x = -r$, then

$$\frac{1}{1+x} = \sum_{k=0}^{n}(-1)^k x^k + \frac{(-1)^{n+1}x^{n+1}}{1+x} = p_n(x) + R_n(x)$$

Show $\lim_{n\to\infty} R_n(x) = 0$. Then

$$\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt = \int_0^x \left(\sum_{k=0}^{n}(-1)^k t^k\right) + \frac{(-1)^{n+1}t^{n+1}}{1+t}\,dt$$

$$= \sum_{k=0}^{n}\int_0^x (-1)^k t^k\,dt + \int_0^x \frac{(-1)^{n+1}t^{n+1}}{1+t}\,dt = \sum_{k=0}^{n}(-1)^k \frac{x^{k+1}}{k+1} + (-1)^{n+1}\int_0^x \frac{t^{n+1}}{1+t}\,dt$$

Does the right term go to 0?

Case 1: $0 \leq x < 1$ $(0 \leq t \leq x < 1)$. Then

$$\int_0^x \frac{t^{n+1}}{1+t}\,dt \leq \int_0^x t^{n+1}\,dt = \frac{x^{n+2}}{n+2} \to 0$$

Case 2: $-1 < x < 0$. Then,

$$x \leq t \leq 0 \implies \frac{1}{1+t} \leq \frac{1}{1+x} \implies \int_0^x \frac{t^{n+1}}{1+t}\,dt \leq \int_0^x \frac{t^{n+1}}{1+x}\,dt = \frac{1}{1+x}\frac{x^{n+2}}{n+2} \to 0$$

Thus,

$$\ln(1+x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{k+1}}{k+1}$$

33.1 Section 8.6

Let

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show $f^{(n)}(0) = 0$ for $n = 0, 1, 2$.

As $|x| \to \infty$, then $e^{-1/x^2} \to e^0 = 1$, and the graph of f is symmetric with respect to the y axis.
Steps, 1. For constant $c > 0$, $e^c = 1 + c + \frac{c^2}{2!} + \cdots + \frac{c^n}{n!} + \cdots > \frac{c^n}{n!}$.
2. Let $x \neq 0$, $n > 0$. Then $e^{1/x^2} > \frac{(1/x^2)^n}{n!} = \frac{1}{n!x^{2n}}$, so $e^{-1/x^2} < n!x^{2n}$.
3. If $k < 2n$, then $\lim_{x\to 0} \frac{e^{-1/x^2}}{|x|^k} < \lim_{x\to 0} \frac{n!x^{2n}}{|x|^k} = \lim_{x\to 0} n!x^{2n-k} = 0$.
4. Let $k = 1, n = 1$, so $k < 2n$. Then $|f'(0)| = \lim_{x\to 0} \left| \frac{f(x)-f(0)}{x-0} \right| = \lim_{x\to 0} \frac{e^{-1/x^2}}{|x|} = 0$ by 3.
5. Use chain rule to find $f'(x)$, $x \neq 0$. $f'(x) = \frac{2}{x^3} e^{-1/x^2}$.
6. Let $k = 4$, $n = 3$ in 3: Then $f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} \frac{f'(x)}{x} = \lim_{x\to 0} \frac{2}{x^4} e^{-1/x^2} = 2 \lim_{x\to 0} \frac{e^{-1/x^2}}{x^4}$
$\leq 2 \lim_{x\to 0} 3!|x|^{6-4} = 0$
7. One can similarly show that $f^{(k)}(0) = 0$ for $k \geq 0$. Then for every taylor polynomial p_n of f about 0 is just $p_n(x) = 0$ for all x.