32 Lagrange Remainder Theorem

Lagrange Remainder Theorem: Let $f^{(n+1)}(x)$ exist on open I, let x_0 be in I. Then,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} = p_n(x) + R_n(x)$$

for x in I, c_x between x and x_0.

Example 32.1 (Exercise 8.2.11)
Assume $f^{(n+1)}$ is continuous on open interval I, and x_0 is in I, and $f^{(k)}(x_0) = 0$ for $k = 1, \cdots, n$ and $f^{(n+1)}(x_0) \neq 0$.

1. If $n+1$ is even and $f^{(n+1)}(x_0) > 0$, then $R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x-x_0)^{n+1} \geq 0$, so $f(x) = f(x_0) + R_n(x) \geq f(x_0)$. for x near x_0. Then x_0 is a local minimizer.
2. If $n+1$ is even and $f^{(n+1)}(x_0) < 0$, then same problem.
3. If $n+1$ is odd, then $R_n(x)$ changes sign at x_0.

Lemma 32.2
Let c be real. Then $\lim_{n\to\infty} \frac{c^n}{n!} = 0$, with $c \neq 0$.

Proof. Let $a_n = \frac{c^n}{n!}$ for $n \geq 1$. Then $\frac{a_{n+1}}{a_n} = \frac{|c|}{n+1} \to 0$. □

Theorem 32.3 (Thm 8.14)
Let $f^{(n)}(x)$ for $n \geq 1$ exist on open interval I, with x_0 in I. If there is an $r > 0$ and an $M < \infty$ so that for x such that $|x - x_0| < r$, we have

$$|f^{(n)}(x)| \leq M^n$$

for all x in I and all $n \geq 0$, then $\lim_{n\to\infty} R_n(x) = 0$, so

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

Proof. By Lagrange Remainder Theorem, $|R_n(x)| = \frac{|f^{(n+1)}(c_x)|}{(n+1)!}|x-x_0|^{n+1}$.
If $|x - x_0| < r$ and x in I, then by hypothesis,

$$|R_n(x)| \leq \frac{M^{n+1}r^{n+1}}{(n+1)!} \to 0$$

Then

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} p_n(x) + \lim_{n\to\infty} R_n(x) = \lim_{n\to\infty} p_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

□

Example 32.4
$f(x) = e^x$, $x_0 = 0$, $I = (-\infty, \infty)$. If $r > 0$ is arbitrary, and $M = \max(1, e^r)$, then

$$|x| < r \implies |f^{(n+1)}(c_x)| = e^{c_x} \leq e^r \leq M \leq M^{n+1}$$

So, by Thm 8.14,

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Example 32.5
$g(x) = \sin x$, $x_0 = 0$. Then $|g^{(n+1)}(x)| \le 1$, for all x, all $n \ge 0$.
So by theorem 8.14,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Example 32.6
$f(x) = \frac{1}{1-x}$, $x_0 = 0 \implies f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f^{(n)}(x) = \frac{k!}{(1-x)^{k+1}}$.

Then $f^{(k)}(0) = k!$. Thus, $\frac{1}{1-x} = \sum_{k=0}^{n} \frac{k! x^k}{k!} = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x} \to \frac{1}{1-x}$ for $|x| < 1$.