31 Taylor Polynomials, Lagrange Remainder Theorem

Recall Proposition 8.2: Let x_0 be in an open interval I, $f : I \to \mathbb{R}$ with n derivatives at x_0. Then there is a unique polynomial p_n of degree $\leq n$ with contact of order n with f at x_0:

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Example 31.1
$h(x) = \ln(1 + x)$ for $x > 1$. Find p_5 with $x_0 = 0$.

Solution: $h(x) = \ln(1 + x)$, $h'(x) = \frac{1}{1+x}$, $h''(x) = \frac{-1}{(1+x)^2}$, $h^{(3)} = \frac{2}{(1+x)^3}$, $h^{(4)}(x) = \frac{3!}{(1+x)^4}$, $h^{(5)}(x) = \frac{4!}{(1+x)^5}$

$h(0) = 0$, $h'(0) = 1$, $h''(0) = -1$, $h^{(3)}(0) = 2$, $h^{(4)}(0) = -3!$, $h^{(5)}(0) = 4!$.
So,

$$p_5(x) = x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3!}{4!}x^4 + \frac{4!}{5!}x^5 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

Note: let $g(x) = \ln(1 - x)$ for $x < 1$. Find p_5 if $x_0 = 0$.
$g'(0) = \frac{-1}{1-x}$, $g''(0) = \frac{-1}{(1-x)^2}$, $\cdots$

$p_5(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$

Example 31.2
$f(x) = e^x$, $x_0 = 0$. Find $p_n(x)$.

Solution: $f^{(k)}(x) = e^x \implies f^{(k)}(0) = 1$ for $k \geq 0$.

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^{(n)}(0)}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

How about $f_1(x) = e^{2x}$, $x_0 = 0$? Find $p_n(x)$.
Solution: $f^{(k)}(x) = 2^k e^{2x}$, so $f^{(k)}(0) = 2^k$. So $p_n(x) = 1 + 2x + \frac{2^2}{2!}x^2 + \cdots + \frac{2^n}{n!}x^n$.

Recall if $f(x) = \sin x$, $x_0 = 0$, then $f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$.

Example 31.3
$g(x) = \cos x$, $x_0 = 0$. Find $p_8(x)$.

Solution: $g'(x) = -\sin x$, $g''(x) = -\cos x$, $g^{(3)}(x) = \sin x$, $g^{(4)}(x) = \cos x$.
$g^{(k)}(0) = 1, 0, -1, 0, \cdots$

$$p_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$$

Note 31.4
Recall the Cauchy MVT (Thm 4.23).

Let f, g be continuous functions on $[a, b]$ to $\mathbb{R}$, and differentiable on (a, b).
Let $g(a) \neq g(b)$ and $g'(x) \neq 0$ for $a < x < b$. Then there is x_0 in (a, b) with

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

Lemma 31.5 (Like Thm 4.24)
Let $f^{(n+1)}$ exist on an open interval I, and x_0 in I. Assume $f^{(k)}(x_0) = 0$ for $k = 0, 1, 2, \cdots, n$ and $f^{(n+1)}(x_0) \neq 0$. Then x in I, $x \neq x_0$ implies there is a c_x between x and x_0 so that

$$f(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}$$

Theorem 31.6 (Lagrange Remainder Theorem (8.8) **)
Let $f^{(n+1)}$ exist on a neighborhood of x_0 in the open interval I. Then

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}$$

Where $R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}$ is the nth Taylor remainder for f.

Proof. Note that $f(x) - p_n(x)$ has kth derivative $= 0$ at x_0 for $k = 0, 1, \cdots, n$ (because f and p_n have contact of order n), and note that $p_n^{(n+1)}(x) = 0$ for all x because p_n is an nth degree polynomial.

So by the lemma, $f(x) - p_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}$ □

Note 31.7
From the Lagrange Remainder Theorem, $f(x) = p_n(x) + R_n(x)$. If $R_n(x) < 0$, then $p_n(x) > f(x)$. If $R_n(x) > 0$, then $p_n(x) < f(x)$.

Example 31.8
If $f(x) = e^x$, $x_0 = 0$, then by the Lagrange Remainder Theorem,

$$f(x) = p_n(x) + R_n(x) = \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right) + \frac{e^{c_x}}{(n+1)!}x^{n+1}$$

Note that the remainder goes to 0 as n gets large, because factorials grow faster than exponentials.

Example 31.9
e is irrational.

Solution: Assume $e = \frac{m}{n}$ for m, n integers, $n \geq 2$ to get contradiction.

Then

$$0 = \frac{m}{n} - \left[\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right) + \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right)\right]$$

Multiplying by $n!$:

$$0 = \left(\frac{m}{n}n!\right) - n!\left[1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right] - (n!)\left[\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right]$$

Everything here is an integer except $(n!)\left[\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots\right]$. Contradiction.

Example 31.10
$f(x) = \sin x \implies$ derivatives at any x are: $|f^{(n+1)}(c_x)| \leq 1$ since $|\pm \sin c_x| \leq 1$, $|\pm \cos c_x| \leq 1$.

So, $|R_n(x)| = \frac{|f^{(n+1)}(c_x)|}{(n+1)!}|x|^{n+1} \leq \frac{|x|^{n+1}}{(n+1)!} \to 0$.