30 Taylor Polynomials

Definition 30.1
Let I be an open interval, x_0 in I. Let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$.

Then f and g have contact of order n if f, g have n derivatives at x_0, and $f^{(k)}(x_0) = g^{(k)}(x_0)$, for $k = 1, \cdots, n$.

Note 30.2
When we say f and g have contact order n, then $f^{(n+1)}(x_0) \neq g^{(n+1)}(x_0)$.

f and g have contact of order 0 if $f(x_0) = g(x_0)$, but $f'(x_0) \neq g'(x_0)$.

f and g have contact of order 1 if $f(x_0) = g(x_0)$, $f'(x_0) = g'(x_0)$.

Example 30.3
$f(x) = x^3 + x$, $g(x) = \sin x$, x real. Find contact of order n with $x_0 = 0$.

Solution: $f(x) = x^3 + x$, $f'(x) = 3x^2 + 1$, $f''(x) = 6x$, $f^{(3)}(x) = 6$.
$g(x) = \sin x$, $g'(x) = \cos x$, $g''(x) = -\sin x$, $g^3(x) = -\cos x$

So $f(0) = 0 = g(0)$, $f'(0) = 1 = g'(0)$, $f''(0) = 0 = g''(0)$, $f^{(3)}(0) = 6 \neq -1 = g^{(3)}(0)$.

So we have contact of order 2.

Example 30.4
$f(x) = 1$, $g(x) = \cos x$ for all x. Find the contact of order n at $x_0 = 0$.

Solution: $f(x) = 1 \implies f'(x) = 0 = f''(x)$
$g(x) = \cos x$, $g'(x) = -\sin x$, $g''(x) = -\cos x$.

Then $f(0) = 1 = g(0)$, $f'(0) = 0 = g'(0)$, $f''(0) = 0 \neq -1 = g''(0)$. So contact of order is 1.

Proposition 30.5 (Prop 8.2)
Let x_0 be in an open interval I, and $f : I \to \mathbb{R}$ with n derivatives at x_0.

Then there is a unique polynomial p_n of degree $\leq n$, with

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

For x near x_0, x in I.
Then $p_n(x)$ is the nth taylor polynomial for f.

Proof. Let $x_0 = 0$ for simplicity. Then to prove $p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$.

Let $p_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$. Show $c_k = \frac{f^{(k)}(0)}{k!}$, $k = 0, 1, \cdots, n$.
Note that $p_n(0) = c_0 = f(0)$.
Next, $p'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1}$. Then $p_n'(0) = c_1 = f'(0)$.
Next, $p''(x) = 2c_2 + 3!c_3 x + 4(3)c_4 x^2 + \cdots n(n-1)c_n x^{n-2}$. Then $p_n''(0) = 2c_2 = f''(0)$, so $c_2 = \frac{f''(0)}{2!}$.

Etc.
$p^{(k)}(0) = \frac{f^{(k)}(0)}{k!}$ for $k = 0, 1, \cdots n$. This yields the desired formula for p_n. □

Example 30.6
$f(x) = \sin x$, $x_0 = 0$. Find p_2.

Solution: $f(x) = \sin x$, $f'(x) = \cos x$, $f''(x) = -\sin x$.
$f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$. So $p_2(x) = 0 + x + 0 = x$.

Example 30.7
$f(x) = 3 + 2x - x^2 = -x^2 + 2x + 3$. Find p_2, p_3 if $x_0 = 4$.

Solution: $f(x) - x^2 + 2x + 3$, $f'(x) = -2x + 2$, $f''(x) = -2$, $f^{(3)}(x) = 0$.
$f(4) = -5$, $f'(4) = -6$, $f''(2) = -2$, $f^{(3)}(2) = 0$.

$$p_2(x) = -5 - 6(x-4) - \frac{2}{2!}(x-4)^2$$

Example 30.8
$f(x) = \sin x$, $x_0 = 0$. Find p_7, p_8, p_{99}.

Solution: $f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$, $f^{(4)}(x) = \sin x$.
Derivatives of f for $n = 0, 1, 2, \cdots$ is $0, 1, 0, -1, 0, 1, 0, -1, \cdots$.

$$p_7(x) = 0 + x + 0 + -\frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + 0 - \frac{1}{7!}x^7$$

Note that $p_7 = p_8$,

$$p_{99}(x) = 0 + \cdots - \frac{1}{99!}x^{99}$$

Example 30.9
$h(x) = \ln(1+x)$, $x > 0$. Find p_5 with $x_0 = 0$.

Solution: $h(x) = \ln(1+x)$, $h'(x) = \frac{1}{1+x}$, $h''(x) = \frac{-1}{(1+x)^2}$, $h^{(3)}(x) = \frac{2}{(1+x)^2}$, $h^{(4)}(x) = \frac{-3!}{(1+x)^4}$, $h^{(5)}(x) = \frac{4!}{(1+x)^5}$.

$h(0) = 0$, $h'(0) = 1$, $h''(0) = -1$, $h^{(3)}(0) = 2$, $h^{(4)}(0) = -3!$, $h^{(5)}(0) = 4!$.

$$p_5(x) = 0 + x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3 - \frac{3!}{4!}x^4 + \frac{4!}{5!}x^5 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$