3 Sequences

3.1 Sequences

Definition 3.1

A **sequence** is a function f defined on all integers $n \geq n_0$ (usually $n_0 = 0$ or 1).
We write $\{a_n\}_{n=n_0}^{\infty}$ where $f(n) = a_n$, for $n \geq n_0$.
Often, we just write $\{a_n\}$, if the n_0 is clear.

Note that n in a_n is an **index**.
The indices of $\{a_n\}_{n=3}^{\infty}$ are $3, 4, 5, \cdots$.

Example 3.2

$\{(-1)^n\}_{n=1}^{\infty}$: $-1, 1, -1, 1, \cdots$
$\{1 - \frac{1}{n}\}_{n=1}^{\infty}$: $0, \frac{1}{2}, \frac{2}{3}, \cdots$
$\{e^n\}_{n=1}^{\infty}$

Definition 3.3

A sequence $\{a_n\}_{n=1}^{\infty}$ is **recursive** or **inductive** if a_1 is given, and $a_{n+1} = f(a_n)$ for $n \geq 1$.

Example 3.4

Suppose we have the sequence $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + \sqrt{2}}$, $\sqrt{2 + \sqrt{2 + \sqrt{2}}}, \cdots$
Here we can see that, $a_2 = \sqrt{2 + a_1}$, and $a_3 = \sqrt{2 + a_2}$
So we can define the sequence as, $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

Definition 3.5

$\{a_n\}_{n=n_0}^{\infty}$ **converges** to **number** L if for each $\epsilon > 0$, there is N^* or N_ϵ so that $n \geq N^* \implies |a_n - L| < \epsilon$.
Then, we write $\lim_{n\to\infty} a_n = L$, and often we will write $a_n \to L$.

Otherwise, if no such L exists, then $\{a_n\}_{n=n_0}^{\infty}$ **diverges**.

Note that there are two types of divergence.
$\{(-1)^n\}_{n=1}^{\infty}$ "wobbles".
$\{n^2\}_{n=1}^{\infty}$. Then, $\lim_{n\to\infty} n^2 = \infty$.

Note 3.6

1. $\lim_{n\to\infty} a_n = a$ if and only if $\lim_{n\to\infty}(a_n - a) = 0$
2. Proposition 2.6: $\lim_{n\to\infty} \frac{1}{n} = 0$ (the sequence $\{\frac{1}{n}\}$ converges to 0)

   Proof. To show that $\{\frac{1}{n}\}$ converges to 0, we must find a n_ϵ such that for any ϵ, $n \geq n_\epsilon$,

   $$\left|\frac{1}{n} - 0\right| < \epsilon \implies \frac{1}{n} < \epsilon$$

   Let $\epsilon > 0$ be arbitrary. By the Archimedean Property, there is $n_\epsilon > 0$ such that $0 < \frac{1}{n_\epsilon} < \epsilon$.
   Then, $n \geq n_\epsilon \implies 0 < \frac{1}{n} < \frac{1}{n_\epsilon} < \epsilon$.
   Thus, $\{\frac{1}{n}\}$ converges to 0, and so $\lim_{n\to\infty} \frac{1}{n} = 0$. □
3. $\lim_{n\to\infty} a_n = \infty$ if for any number $M > 0$, there is an n_M so that $n \geq n_M \implies a_n > M$.

3.2 Properties of convergence

Property 3.7 (Comparison Property - Lemma 2.9)
Assume that $a_n \to a$, and $\{b_n\}_{n=1}^{\infty}$ is a sequence, and b is a number.
If $c \geq 0$ is a number so that $|b_n - b| \leq c|a_n - a|$, for all $n \geq n^*$, then $b_n \to b$.

Proof. Note $n \geq n^* \implies |b_n - b| \leq c|a_n - a| \xrightarrow{n} 0$. From this, it follows that $b_n - b \to 0$, and so $b_n \to b$, meaning the series converges. □

Example 3.8
Show that $\lim_{n\to\infty} \frac{1}{n^2+n} = 0$.

Solution: for $n \geq 1$, $0 < \frac{1}{n^2+n} \leq \frac{1}{n} \xrightarrow{n} 0$ by Prop 2.6. So by the Comparison Property, $\frac{1}{n^2+n} \xrightarrow{n} 0$

Definition 3.9
$\{a_n\}_{n=n_0}^{\infty}$ is **bounded** if there is some number M so that $|a_n| \leq M$ for all n.

Theorem 3.10 (Theorem 2.18 **)
If $\{a_n\}_{n=n_0}^{\infty}$ converges, then it is bounded.

Proof. Let $\epsilon > 0$ be arbitrary. If $a_n \to a$, then there is an N^* so that $n \geq N^* \implies |a_n - a| < \epsilon$.
Let $M = \max\{|a_1|, |a_2|, \cdots, |a_{N^*}|, |a| + \epsilon\}$. Then we have that $|a_n| \leq M$ for all $n \geq 0$. □

Theorem 3.11 (Sum Rule)
If $a_n \to a$, $b_n \to b$, then $a_n + b_n \to a + b$.

Proof. $|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| \to 0$, because $|a_n - a| \to 0$ and $|b_n - b| \to 0$.

Alternatively, we know that there must be some indices n_1, n_2 for sequences $\{a_n\}, \{b_n\}$ respectively such that $|a_n - a| < \frac{\epsilon}{2}$ for $n \geq n_1$, and $|b_n - b| < \frac{\epsilon}{2}$ for $n \geq n_2$. So, we can take $m = \max\{n_1, n_2\}$ as our index such that for all $n \geq m$,

$$|a_n + b_n - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And thus $\{a_n + b_n\}$ converges to $a + b$. □

Theorem 3.12 (Product Rule)
If $a_n \to a$, $b_n \to b$, then $a_n b_n \to ab$.

Proof. $\lim_{n\to\infty} b_n = b$ implies, by Thm 2.18, $|b_n| \leq M$ for all $n \geq n_0$.
Then, $|a_n b_n - ab| \leq |a_n b_n - a b_n| + |a b_n - ab| = |a_n - a||b_n| + |a||b_n - b| \leq M|a_n - a| + |a||b_n - b| \to 0$, so $a_n b_n \to ab$, where the first inequality is created by the triangle inequality and adding an extra $(ab_n - ab_n)$. □

Theorem 3.13 (Quotient Rule)
If $a_n \to a$, $b_n \to b$, then if $b \neq 0$, then $\frac{a_n}{b_n} \to \frac{a}{b}$.

Note 3.14

Suppose we want to prove $\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2})$ exists and find it.
We would multiply by the conjugate as follows:

$$\begin{aligned}
\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2}) &= \lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2})\left(\frac{\sqrt{n^2+n}+\sqrt{n^2}}{\sqrt{n^2+n}+\sqrt{n^2}}\right) \\
&= \lim_{n\to\infty}\frac{(n^2+n)-n^2}{\sqrt{n^2+n}+\sqrt{n^2}} \\
&= \lim_{n\to\infty}\frac{n}{\sqrt{n^2}+\sqrt{n^2}} \\
&= \lim_{n\to\infty}\frac{n}{n+n} \\
&= \frac{1}{2}
\end{aligned}$$