29 Trapezoidal Rule, Simpson's Rule

29.1 Trapezoidal Rule

Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $P_n = \{a = x_0, x_1, \cdots, x_n = b\}$ be a regular partition with gap$P_n = \frac{b-a}{n}$.

For the interval $[x_{i-1}, x_i]$, the mean value theorem for integrals states that for an $x^* \in [x_{i-1}, x_i]$.

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = f(x^*)\frac{b-a}{n} \approx \frac{f(x_{i-1}) - f(x_i)}{2}\frac{b-a}{n}$$

This leads to

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left([f(a) + f(x_1)] + [f(x_1) + f(x_2)] + \cdots + [f(x_{n-1}) + f(b)]\right)$$

And thus, we get the Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}[f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(b)]$$

Example 29.1
Approximate $\int_0^2 x^3\,dx$ by the Trapezoidal Rule with $n = 6$.

Solution:

$$\int_0^2 x^3\,dx \approx \frac{2-0}{2\cdot 6}\left[0^3 + 2\left(\frac{1}{3}\right)^3 + 2\left(\frac{2}{3}\right)^3 + \cdots + 2\left(\frac{5}{3}\right)^3 + 2^3\right]$$

Note 29.2
Notes about Trapezoidal Rule:

1. If $g : [a, b] \to \mathbb{R}$ is linear, then $\int_a^b g(x)\,dx =$ trapezoidal rule.
2. If the graph of g is concave upwards on $[a, b]$, then $\int_a^b g(x)\,dx <$ trapezoidal rule.
3. Similarly, if the graph is concave downwards, then $\int_a^b g(x)\,dx >$ trapezoidal rule.
4. The trapezoidal rule is effective for approximating $\int_a^b$ if $\int_a^b f - \text{T.R.} \approx 0$.

Let E_n^T be the error in approximating $\int_a^b$ by the Trapezoidal Rule with a partition of n subintervals.

If $f''(x)$ exists for $a < x < b$, then it turns out that

$$E_n^T \le \frac{M_T}{12n^2}(b-a)^3$$

where $M_T = \sup\{|f''(x)| : a < x < b\}$

Example 29.3
Let $f(x) = x^3$, $0 \le x \le 2$. Find a reasonable n so $E_n^T \le \frac{1}{100}$.

Solution: $f'(x) = 3x^2$, $f''(x) = 6x$. For $0 < x < 2$, $|f''(x)| < |6x| \le 12 = M_T$.

Then

$$E_n^T \le \frac{12}{12n^2}(2^3) = \frac{8}{n^2} \le \frac{1}{100} \text{ if } 800 \le n^2 \text{ if } n = 29$$

29.2 Simpson's Rule

We use parabolas to approximate the graph of f on $[a, b]$, with f continuous.
Let $P_n = \{a = x_0, x_1, \cdots, x_n = b\}$ be regular, with gap$P_n = \frac{b-a}{n}$, and let n be positive and even.

For $a = x_0 < x_1 < x_2 \leq b$, let

$$p(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{h}(x - x_0) + \frac{f(x_0) - 2f(x_1) + f(x_2)}{2h^2}(x - x_0)(x - x_1)$$

Where $h = \frac{b-a}{n}$.

Then p is quadratic, and $p(x_0) = f(x_0)$, $p(x_1) = f(x_1)$, and $p(x_2) = f(x_2)$. We create the unique parabola that goes through these 3 points.

Then,

$$\int_{x_0}^{x_2} p(x)\, dx = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

This leads to:

$$\int_a^b f(x)\, dx \approx \frac{b-a}{3n}[[f(x_0) + 4f(x_1) + f(x_2)] + [f(x_2) + 4f(x_3) + f(x_4)] + \cdots + [f(x_{n-2}) + 4f(x_{n-1}) + f(b)]]$$

And thus we get Simpson's Rule:

$$\int_a^b f(x)\, dx = \frac{b-a}{3n}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(b)\right]$$

Example 29.4
Approximate $\int_0^2 x^5\, dx$ by Simpson's Rule, $n = 4$.

Solution:

$$\int_0^2 x^5\, dx \approx \frac{2-0}{3(4)}\left[0^5 + 4\left(\frac{1}{2}\right)^5 + 2\left(\frac{2}{2}\right)^5 + 4\left(\frac{3}{2}\right)^5 + 2^5\right] = 10.75$$

Simpson's Error:

$$E_n^S \leq \frac{M_S(b-a)^5}{180n^4}$$

where $M_S = \sup\{|f^{(4)}(x)| : a < x < b\}$.

Example 29.5
$f(x) = x^3 \implies f^{(4)}(x) = 0 \implies M_S = 0 \implies E_n^S = 0.$