28 Exam Review, Integration by Parts

28.1 Exam Review

Example 28.1 (Exam 2a)
$f'(x_0)$ exists implies that f is continuous at x_0

$$\lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}(x-x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\lim_{x\to x_0}(x-x_0) = 0$$

Thus, f is continuous at x_0.

Example 28.2 (Exam 3a)
Identity Criterion: f', g' exist on open interval I. Then $f' = g'$ on $I \iff$ there is a C so $f(x) = g(x) + C$ for all x in I.

Because of the identity criterion, one only needs 1 antiderivative in integration.

Example 28.3 (Exam 3b)
The area between the graph of $\sin x$ on $[-\frac{\pi}{3}, \frac{\pi}{2}]$ and the x axis:

$$A = \int_{-\pi/3}^{0} -\sin x\,dx + \int_0^{\pi/2} \sin x$$

Example 28.4 (Exam 4c)
$f : [2,3] \to \mathbb{R}$ is decreasing. Show f is integrable using Archimedes Riemann Theorem.

<u>Proof:</u> Let $(P_n)_{n=1}^{\infty}$ be a sequence of regular partitions of $[2,3]$ with gap$P_n = \frac{1}{n}$.

Then $U(f,P_n) - L(f,P_n) = \sum_{i=1}^{n} f(x_{i-1})\frac{1}{n} - \sum_{i=1}^{n} f(x_i)\frac{1}{n} = \frac{1}{n}(f(2) - f(3)) \to 0$ (Use Archimedes Riemann Theorem)

Example 28.5 (Exam 5b)

$$\begin{aligned}\frac{d}{dx}\int_{x^2}^{2}\frac{\sin x}{t}\,dt &= \frac{d}{dx}\left[-(\sin x)\int_2^{x^2}\frac{1}{t}\,dt\right]\\ &= -\cos x\int_2^{x^2}\frac{1}{t}\,dt - \sin x\frac{d}{dx}\int_2^{x^2}\frac{1}{t}\,dt\\ &= (-\cos x)(\ln x^2 - \ln 2) - (\sin x)\frac{1}{x^2}(2x)\end{aligned}$$

28.2 Integration by Parts

Theorem 28.6 (Thm 7.5)
Let $g : [a,b] \to \mathbb{R}$, $h : [a,b] \to \mathbb{R}$ be continuous, with continuous derivatives on (a,b).
Then,

$$\int_a^b h(x)g'(x)\,dx = h(b)g(b) - h(a)g(a) - \int_a^b h'(x)g(x)\,dx$$

Proof.

$$\begin{aligned}
\int_a^b (h(x)g'(x) + h'(x)g(x))\,dx &= \int_a^b \frac{d}{dx}(h(x)g(x))\,dx \\
&= h(x)g(x)\Big|_a^b \\
&= h(b)g(b) - h(a)g(a) \\
\int_a^b (h(x)g'(x) + h'(x)g(x))\,dx &= \int_a^b h(x)g'(x)\,dx + \int_a^b h'(x)g(x)\,dx \\
&= h(b)g(b) - h(a)g(a)
\end{aligned}$$

So,

$$\int_a^b h(x)g'(x)\,dx = h(b)g(b) - h(a)g(a) - \int_a^b h'(x)g(x)\,dx$$

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Similarly, if $u = h(x)$ and $dv = g'(x)\,dx$, then

$$\int u\,dv = uv - \int v\,du$$

Example 28.7

$$\int x \sin x\,dx$$

Here, we let $u = x$, and $dv = \sin x\,dx$.

$$\int x\sin x\,dx = x(-\cos x) + \int \cos x\,dx = -x\cos x + \sin x + C$$

Example 28.8

$$\int \ln x\,dx = \int 1 \ln x\,dx$$

Let $u = \ln x$, $dv = 1\,dx$

$$\int 1\ln x\,dx = x\ln x - \int x\frac{1}{x}\,dx = x\ln x - x + C$$

Example 28.9

$$\tan^{-1} x = \int 1 \tan^{-1} x\,dx$$

$u = \tan^{-1} x,\ dv = 1$

$$\int 1\tan^{-1} x\,dx = x\tan^{-1} x - \int x\frac{1}{1+x^2}\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C$$

Example 28.10

$$\int e^{2x} \sin x$$

Let $u = e^{2x}$, $dv = \sin x\, dx$

$$\int e^{2x} \sin x = e^{2x}(-\cos x) + \int 2e^{2x} \cos x\, dx$$

Let $u = 2e^{2x}$, $dv = \cos x\, dx$

$$e^{2x}(-\cos x) + \int 2e^{2x} \cos x\, dx = -e^{2x} \cos x + 2e^{2x} \sin x - \int 4e^{2x} \sin x\, dx$$

So, we have that

$$5 \int e^{2x} \sin x\, dx = (-e^{2x} \cos x + 2e^{2x} \sin x) + C$$

So

$$\int e^{2x} \sin x\, dx = \frac{1}{5}(-e^{2x} \cos x + 2e^{2x} \sin x) + C$$

Theorem 28.11 (u-substitution)
Let $f : [a, b] \to \mathbb{R}$, $g : [c, d] \to \mathbb{R}$ be continuous, and g' be bounded and continuous.

Assume $g(c, d) \subseteq (a, b)$. Then

$$u = g(x) \implies \int_c^d f(g(x))g'(x)\, dx = \int_{g(c)}^{g(d)} f(u)\, du$$

Example 28.12
Using the substitution $u = 1 - x$, $x = 1 - u$, $du = -dx$,

$$\begin{aligned} \int x\sqrt{1-x}\, dx &= \int (1-u)\sqrt{u}(-1)\, du \\ &= \int (u^{3/2} - u^{1/2})\, du \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{5}(1-u)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C \end{aligned}$$

Example 28.13
With the substitution $u = \cos(5t)$, $du = -5\sin(5t)\, dt$,

$$\int_0^{\pi/2} \cos^3(5t) \sin(5t)\, dt = \int_1^0 \frac{-1}{5} u^3\, du = -\frac{1}{5} \cdot \frac{1}{4} u^4 \Big|_1^0 = \frac{1}{20}$$