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Example 26.1 (Problem 6.6.4)
Let

$$g(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)\,dt = f(0) + f'(0)x + x\int_0^x f''\,dt - \int_0^x tf''(t)\,dt$$

Then

$$g'(x) = f'(0) + \int_0^x f''(t)\,dt + xf''(x) - xf''(x) = f'(0) + \int_0^x f''(t)\,dt$$

$$g''(x) = f''(x)$$

for all x. Then by the identity criterion, there is C_1 with $g'(x) = f'(x) + C_1$ for all x.

$$g'(0) = f'(0) + 0 = f'(0) \implies g'(x) = f'(x) \qquad \forall x$$

So, by the identity criterion, there is C_2 so that $g(x) = f(x) + C_2$ for all x.

$$g(0) = f(0) + 0 + 0$$

So

$$g(x) = f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)\,dt$$

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1. (a)

$$f(x) = \frac{1}{2+x}$$

$$f'(2) = \lim_{x\to 2}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2}\frac{\frac{1}{2+x}-\frac{1}{4}}{x-2} = \lim_{x\to 2}\frac{4-(2+x)}{(2+x)4(x-2)} = \lim_{x\to 2}\frac{-1}{4(2+x)} = \frac{-1}{16}$$

   (b) $g(x) = x^5 + 5x + 3$ for all x. Show g^{-1} exists and find $(g^{-1})'(9)$

   Solution: $g'(x) = 5x^4 + 5$ for all x means g is strictly increasing, so g^{-1} exists.
   Note $g(1) = 9$. So, $(g^{-1})'(9) = \frac{1}{g'(1)} = \frac{1}{10}$.

2. (a) f has exactly two local maximizers, x_1 and x_2, with $x_1 < x_2$.

   On $[x_1, x_2]$, h has a max and min value by the extreme value theorem. because continuous.
   Say the minimum value is $h(z^*)$. Then z^* is a local minimizer of h.

   (b) Assume $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and M is anumber such that $|f'(x)| \le M$ for all real x. Prove that f is uniformly continuous.

   Solution: note f is uniformly continuous if for any $\{u_n\}, \{v_n\} \subseteq D$, with $|u_n - v_n| \to 0$, then $|f(u_n) - f(v_n)| \to 0$.

   Let $\{u_n\}, \{v_n\}$ be arbitrary, and $|u_n - v_n| \to 0$.
   For each n, there is an x_n in interval between u_n and v_n with $\frac{f(u_n)-f(v_n)}{u_n-v_n} = f'(x_n)$ by the mean value theorem.

   Then $|f(u_n) - f(v_n)| = |f'(x_n)||u_n - v_n| \le M|u_n - v_n| \to 0$. So, f is uniformly continuous.

3. (a) Let $g : [a, b] \to \mathbb{R}$ be bounded. Prove that $\underline{\int_a^b} g(x)\,dx$ exists.
   Let P be any partition of $[a, b]$. Then g is bounded, so there is an m so $m \le g(x)$ for all x in $[a, b]$, and there is an M so that $m \le g(x) \le M$ for all x.

   Then, $m(b-a) \le L(f, P) \le M(b-a)$, so $\underline{\int_a^b} f = \sup_P L(f, P)$ exists.

(b) Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = F(b) - F(a)$, and $F' = f$ on (a, b).

Let G be continuous on $[a, b]$, and $G' = f$ on (a, b).

Then by the identity criterion, $G(x) = F(x) + C$ for $a < x < b$. But F is also continuous on $[a, b]$. Then $G(x) = F(x)$ for all x in $[a, b]$. So $G(b) - G(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a)$.

4. (a) 4 properties: area is nonnegative, additive property, rectangle property, comparison property.

   (b) State either 1st or 2nd fundamental theorem, not both.

5. (a)

$$\frac{d}{dx}\int_2^x \frac{3x}{t}\,dt = \frac{d}{dx}3x\int_2^x \frac{1}{t}\,dt = 3\int_2^x \frac{1}{t}\,dt + 3x\left(\frac{1}{x}\right) = 2(\ln x - \ln 2) + 3$$