25 First Fundamental Theorem, Integral IVT, Improper Integrals

25.1 First Fundamental Theorem, Integral IVT

Definition 25.1
$\ln x = \int_1^x \frac{1}{t}\,dt$ for $x > 0$.
Note $\ln 1 = 0$, $\ln(ab) = \ln a + \ln b$ if $a, b > 0$.

Note 25.2
Recall the following:

Thm 6.18: If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable on $[a, b]$.

If f is continuous on $[a, b]$, and $G(x) = \int_a^x f(t)\,dt$, $a \le x \le b$, then $G'(x) = f(x)$ for $a \le x \le b$.

Second Fundamental Theorem ****: If $f : [a, b] \to \mathbb{R}$ is continuous, then $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, $a < x < b$.

Theorem 25.3 (First Fundamental Theorem of Calculus ****)
Let $F : [a, b] \to \mathbb{R}$ be continuous, and $F' = f$ on (a, b), with f continuous and bounded on (a, b). Then $\int_a^b f(t)\,dt = \int_a^b F'(t)\,dt = F(b) - F(a)$.

Proof. Let $G(x) = \int_a^x f(t)\,dt$, for $a \le x \le b$. Then we proved that $G'(x) = f(x)$ for $a < x < b$.

Thus, $G'(x) = f(x) = F'(x)$, for $a < x < b$.

Then by the Identity Criterion, there is a constant C such that $G(x) = F(x) + C$ for $a < x < b$.

$G(a) = 0 \implies \int_a^b f(t)\,dt = G(b) - G(a) = (F(b) + C) - (F(a) + C) = F(b) - F(a)$)since F, G are continuous on $[a, b]$. □

Note 25.4
In effect, by the first fundamental theorem, the integral of the derivative of F is essentially F (differing by a constant)

By the second fundamental theorem, the derivative of the integral of f is f.

By the first and second notes, differentiation and integration are essentially inverse processes.

Note 25.5
How about $\int_0^1 x\sqrt{1 + x^2}\,dx$?

Let $u = 1 + x^2$, so $du = 2x\,dx$, $x = 0 \implies u = 1$, and $x = 1 \implies u = 2$.

$$\int_0^1 x\sqrt{1 + x^2}\,dx = \int_1^2 \frac{1}{2}u^{1/2}\,du$$

How about $\int_1^2 \sqrt{1 + x^4}\,dx$
$G(x) = \int_1^x \sqrt{1 + t^4}\,dt$ is an anti derivative!

Theorem 25.6 (Mean Value Theorem for Integrals - Thm 6.26)
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there is an x_0 in (a, b) such that $\frac{1}{b-a}\int_a^b f(t)\,dt = f(x_0)$, or equivalently, $\int_a^b f(t)\,dt = f(x_0)(b-a)$.

Proof. Let $m = \min_{a\le x\le b} f(x)$, and $M = \max_{a\le x\le b} f(x)$. Then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

So

$$m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M$$

f continuous on $[a, b]$ implies by the IVT that there is an x_0 in (a, b) with $f(x_0) = \frac{1}{b-a}\int_a^b f(t)\,dt$ ☐

Example 25.7 (Exercise 6.6.5)
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and let $G(x) = \int_0^x (x-t)f(t)\,dt$. Show $G''(x) = f(x)$ for all x.

Solution:

$$\begin{aligned}
G(x) &= \int_0^x (x-t)f(t)\,dt \\
&= \int_0^x xf(t)\,dt - \int_0^x tf(t)\,dt \\
&= x\int_0^x f(t)\,dt - \int_0^x tf(t)\,dt \\
G'(x) &= \int_0^x f(t)\,dt + x\frac{d}{dx}\int_0^x f(t)\,dt - \frac{d}{dx}\int_0^x tf(t)\,dt \\
&= \int_0^x f(t)\,dt + xf(x) - xf(x) \\
&= \int_0^x f(t)\,dt \\
G''(x) &= \frac{d}{dx}\int_0^x f(t)\,dt = f(x)
\end{aligned}$$

25.2 Improper Integrals

Definition 25.8
Let f be continuous on $[a, b)$, with $\lim_{x\to b^-} f(x) = \infty$.
If $\lim_{c\to b^-}\int_a^c f(x)\,dx$ is a number T, then $\int_a^b f(x)\,dx = T$, and $\int_a^b f(x)\,dx$ converges.

Otherwise, $\int_a^b f(x)\,dx$ diverges.

Similarly for $\lim_{c\to a^+}\int_a^c f(t)\,dt$, and $\lim_{b\to\infty}\int_a^b f(x)\,dx$, etc.

Example 25.9
Determine if $\int_1^2 \frac{1}{(x-1)^{4/3}}\,dx$ converges or diverges.

Solution:

$$\int_1^2 \frac{1}{(x-1)^{4/3}}\,dx = \lim_{a\to 1^+}\int_a^2 \frac{1}{(x-1)^{4/3}}\,dx = \lim_{a\to 1^+} \frac{-1}{3(x-1)^{1/3}}\Big|_a^2 = \cdots = \infty$$

So, it diverges.