24 Fundamental Theorem of Calculus, Logarithms

Items:

1. If $f : [a, b] \to \mathbb{R}$ is bounded and integrable, then why is $f(a)$ irrelevant to $\int_a^b f$?
2. If g is a step function on $[a, b]$, why is g integrable?

   Solution: g is bounded, so let $|g(x)| \leq M$.
   Let g have k steps. Let P_n be a regular partition with gap$P_n = \frac{b-a}{n}$, with n large.

   Then $U(g, P_n) - L(g, P_n) \leq (k-1) \cdot 2M \cdot \frac{b-a}{n} \to 0$ (the difference is different from 0 when a partition overlaps two different steps)

3. $f(x) = \begin{cases} 0 & x \text{ rational in } [0,1] \\ 1 & x \text{ irrational in } [0,1] \end{cases}$. Then f is bounded but not integrable.

Definition 24.1
If $a < b$, then $\int_a^b f = -\int_b^a f$ by definition, so $\int_b^a f = -\int_a^b f$.

Definition 24.2
If $f : [a, b] \to \mathbb{R}$ is continuous, then define $F(x) = \int_a^x f(t)\, dt$ for $a \leq x \leq b$.

By the linearity, f is integrable on $[a, x]$ for $a \leq x \leq b$.

Theorem 24.3
Let $f : [a, b] \to \mathbb{R}$ be continuous, and $F(x) = \int_a^x f(t)\, dt$.

Then $F'(x) = f(x)$ for $a < x < b$, and F is continuous on $[a, b]$.

Proof. Let x_0 be arbitrary in (a, b), and $x \approx x_0$ with x in (a, b). Then

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\int_a^x f(t)\, dt - \int_a^{x_0} f(t)\, dt}{x - x_0} = \lim_{x \to x_0} \frac{\int_{x_0}^x f(t)\, dt}{x - x_0} = \frac{f(x_0)(x - x_0)}{x - x_0} = f(x_0)$$

So, $F'(x_0) = f(x_0)$, for all x_0 in (a, b). Not at the endpoints because there is no neighborhood around the endpoints.
So, F is continuous on (a, b) (Thm 4.5 states differentiability implies continuity).

To show F is continuous on $[a, b]$, let $x_0 = a$.
Let $x_n \to a$, $a \leq x_n \leq b$. Then

$$|F(x_n) - F(a)| = \left| \int_a^{x_n} f(t)\, dt - \int_a^a f(t)\, dt \right| = \left| \int_a^{x_0} f(t)\, dt \right| \approx f(a)(x_n - a) \to 0$$

So, F is continuous at a.

Similarly, F is continuous at b, so F is continuous on $[a, b]$. □

Theorem 24.4 (Second Fundamental Theorem of Calculus ****)
If $f : [a, b] \to \mathbb{R}$ is continuous, then $\frac{d}{dx}\left[\int_a^x f(t)\, dt\right] = f(x)$ for $a < x < b$.

Proof. Proof is in the preceding theorem. □

Corollary 24.5 (Corollary 6.32)
Let I, J be open intervals, and $f : I \to \mathbb{R}$, $\phi : J \to \mathbb{R}$, with $\phi(J) \leq I$.

Assume f, ϕ are diffferentiable. Then

$$\frac{d}{dx}\int_a^{\phi(x)} f(t)\,dt = (f(\phi(x)))\phi'(x)$$

for x in J.

Proof. Let $G(x) = \int_a^x f(t)\,dt$, x in the domain of G.
By the Chain Rule, if $G(\phi(x)) = \int_a^{\phi(x)} f(t)\,dt$, then

$$\frac{d}{dx}G(\phi(x)) = \frac{d}{dx}\int_a^{\phi(x)} f(t)\,dt = [f(\phi(x))](\phi'(x))$$

□

Example 24.6
$G_1(x) = \int_a^{x^2} \sin t\,dt \implies G_1'(x) = \big(\sin(x^2)\big)\,(2x)$

$G_2(x) = \int_{e^{2x}}^{a} \sin t\,dt = -\int_a^{e^{2x}} \sin t\,dt \implies G_2'(x) = \big(-\sin(e^{2x})\big)\,(2e^{2x})$

$$G_3(x) = \int_0^x \sin(x+t)\,dt = \int_0^x (\sin x\cos t + \cos x\sin t)\;dt = (\sin x)\int_0^x \cos t\,dt + (\cos x)\int_0^x \sin t\,dt$$

$$G_3'(x) = \left[(\cos x)\int_0^x \cos t\,dt + (\sin x)(\cos x)\right] + \left[(-\sin x)\int_0^x \sin t\,dt + (\cos x)(\sin x)\right]$$

Definition 24.7
Let $G(x) = \int_1^x \frac{1}{t}\,dt$ for $x > 0$.

Note that $G(1) = 0$.

Example 24.8
Show $G(ax) = G(a) + G(x)$ for $a > 0, x > 0$.

Proof. Let $H(x) = G(ax) - G(a) - G(x)$, $x > 0$. We show that $H(x) = 0$ for $x > 0$.

$H'(x) = G'(ax) - 0 - \frac{1}{x}$
$G(ax) = \int_1^{ax} \frac{1}{t}\,dt \implies G'(ax) = \frac{1}{ax}\cdot a = \frac{1}{x}$.

So, $H'(x) = \frac{1}{x} - 0 - \frac{1}{x} = 0$ for all $x > 0$.

So, there is a constant C such that $H(x) = C$ for all $x > 0$.

$H(1) = G(a) - G(a) - G(1) = 0$, so $C = 0$.

Thus, $H(x) = 0$ for $x > 0$. □