23 Area, Integrability

Important theorems in Chapter 6 (on integration)

- Archimedes-Riemann Theorem: Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable if and only if there is a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ of $[a, b]$ with $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$.
- Additivity Theorem (6.12): If $f : [a, b] \to \mathbb{R}$ is bounded, and $a < c < b$, and $\int_a^b f$ exists, then so do $\int_a^c f, \int_c^b f$, and $\int_a^b f = \int_a^c f + \int_c^b f$.
- Monotonicity Theorem (6.13): $\int_a^b f \leq \int_a^b g$ if $f \leq g$ on $[a, b]$.
- Linearity (6.15): $\int_a^b(\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$ where α, β are constants.

23.1 Area

Big items:

1. The area A of rectangle of base $b - a$ and height c is $c(b - a) = \int_a^b c$ where $a < b$, $c \geq 0$.
2. Area $A \geq 0$. The area of the region between the graph of f and the x axis on $[a, b]$ is $\int_a^b |f|$.
3. Addition Property: The sum of the areas A_1 and A_2 of 2 non-intersecting regions is the sum of the areas.
4. Comparison Property: A smaller region has less area than a larger region. If $0 \leq f \leq g$ on $[a, b]$, and f, g integrable, then $\int_a^b f \leq \int_a^b g$.

Theorem 23.1 (Thm 6.18 ***)
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable on $[a, b]$ ($\int_a^b f$ exists).

Proof. Let $\epsilon > 0$ be arbitrary. Then note f continuous on $[a, b]$ implies that f is uniformly continuous on $[a, b]$. Then there is $\delta > 0$ so if x, z are in $[a, b]$ and $|x - z| < \delta$, then $|f(x) - f(z)| < \frac{\epsilon}{b-a}$.

By the Archimedes Property, there is an n so $\frac{b-a}{n} < \delta$.
Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of regular partitions of $[a, b]$ with gap$P_n = \frac{b-a}{n}$.

Then $U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n}(M_i - m_i)\frac{b-a}{n} < \sum_{i=1}^{n} < \frac{\epsilon}{b-a} \cdot \frac{b-a}{n} = \epsilon \sum_{i=1}^{n} \frac{1}{n} = \epsilon$.

So, the Archimedes Riemann Theorem implies that f is integrable on $[a, b]$. □

Corollary 23.2 (6.19)
Let $f : [a, b] \to \mathbb{R}$ be bounded and continuous on $(a, b]$. Then f is integrable.

Proof. Let $P_n = \{a = x_0, x^*, x_2, \cdots, x_n = b\}$ be regular, gap$P_n = \frac{b-a}{n}$ for $n \geq 1$.

Since f is continuous on $[x^*, b]$, f is integrable on $[x^*, b]$, so we can assume if $P_n^* = \{x^*, x_2, \cdots, x_n = b\}$, then $U(f, P_n^*) - L(f, P_n^*) \to 0$.

Then $U(f, P_n) - L(f, P_n) = (M_1 - m_1)(x^* - a) + U(f, P_n^*) - L(f, P_n^*) \to 0$ as $n \to \infty$ and $x^* \to a$, because M_1 and m_1 are bounded, and $x^* - a \to 0$. □

Example 23.3
Let $g_n(x) = \cos(2n\pi x)$, $0 \le x \le 1$.

$g_n\left(\frac{1}{n}\right) = 1$, $g_n\left(\frac{2}{n}\right) = \cos 4\pi = 1$

$g_n\left(\frac{k}{n}\right) = 1, g_n\left(\frac{k}{n} + \frac{1}{2n}\right) = -1$

Given $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots\}$, $U(g_n, P_n) - L(g_n, P_n) = \sum_{i=1}^{n}(1-(-1))\frac{1}{n} = 2$.

But g is integrable since it is continuous. This is only a particular P_n.