22 Additivity, Monotonicity, and Linearity of Integrals

Example 22.1 (Problem 6.2.6a)
Show

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}$$

Solution: Let P_n be a regular partition of $[a, b]$, gap $P_n = \frac{b-a}{n}$, $n \geq 1$.

Since $0 \leq a$ by hypothesis, then f is strictly increasing on $[a, b]$, so f is integrable by Example 6.9.

Now,

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} f(x_i) \frac{b-a}{n} \\ &= \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right) \frac{b-a}{n} \\ &= \frac{b-a}{n} \left(na + \frac{b-a}{n} \sum_{i=1}^{n} i\right) \\ &\approx \frac{b-a}{2} \quad \text{as } n \to \infty \end{aligned}$$

Questions from Monday:

1. Is a partition a finite set of points? Yes.
2. Why is $L(f, P) \leq U(f, Q)$, any partitions P, Q of $[a, b]$?

   If P^* is a common refinement, then $L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, Q)$

3. Note: $\sup_P L(f, P) \leq A \leq \inf_Q U(f, Q)$ (there is always a number A with this property)

   If only one A exists, then $A = \int_a^b f$. (because $\sup_P L(f, P) = \underline{\int}_a^b f$, $\inf_Q U(f, Q) = \overline{\int}_a^b$)
4. If $f : [a, b] \to \mathbb{R}$ is monotone, then is f bounded? Yes, if f is monotonically increasing on $[a, b]$, then $f(b)$ is the max value, and $f(a)$ is the minimum value (because we have a closed interval, and extreme value theorem).

Theorem 22.2 (Archimedes-Riemann Theorem)
Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable if and only if there is a sequence $\{P_n\}_{n=1}^{\infty}$ of partitions of $[a, b]$ with $U(f, P_n) - L(f, P_n) \to 0$.

Note 22.3
Note: we usually take P_n to be a regular partition with gap $P_n = \frac{b-a}{n}$.

Note: If $\lim_{x \to a^+} f(x) = \infty$, then there is a crisis with $U(f, P)$ since $\sup_{a \leq x \leq x_1} f(x) = \infty$.

Note: $(x) = x^k$ on $[a, b] \implies$ if $a \geq 0$ or $b \leq 0$, then f is integrable. Then x^k is integrable, since it is monotone.

Theorem 22.4 (Additivity of Integral - Thm 6.12)
Assume $f : [a, b] \to \mathbb{R}$ is integrable, and $a < c < b$.
Then f is integrable on $[a, c]$ and on $[c, b]$.

Proof. Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[a, b]$ with c a pt of P_n for each P_n with $U(f, P_n) - L(f, P_n) \to 0$. Let $Q_n = P_n \cap [a, c]$, and $R_n = P_n \cap [c, b]$.

Then $U(f, Q_n) - L(f, Q_n) \to 0$, and $U(f, R_n) - L(f, R_n) \to 0$. So f is integrable on $[a, c]$ and on $[c, b]$ by Archimedes Riemann Theorem. □

Theorem 22.5 (Thm 6.13 (Monotonicity))
Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ be bounded and integrable.

Assume $f \leq g$ on $[a, b]$. Then $\int_a^b f \leq \int_a^b g$.

Proof. Let $P = \{a = x_0, x_1, \cdots, x_n = b\}$ be a partition of $[a, b]$. Then

$$L(f, P) = \sum_{i=1}^{n} m_i^f (m_i - x_{i-1}) \leq \sum_{i=1}^{n} m_i^g (x_i - x_{i-1}) = L(g, P)$$

And also, $U(f, P) \leq U(g, P)$.

Then, $\sup_P L(f, P) \leq \sup_P L(g, P)$ and $\inf U(f, P) \leq \inf U(g, P)$
So, $\int_a^b f \leq \int_a^b g$. □

Lemma 22.6
If $f : [a, b] \to \mathbb{R}$ is bounded and integrable, and α is constant, then αf is integrable, and $\int_a^b \alpha f = \alpha \int_a^b f$.

Proof.

$$L(\alpha f, P) = \sum_{i=1}^{n} a m_i (x_i - x_{i-1}) = \alpha \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \alpha L(f, P)$$

Similarly, $U(\alpha f, P) = \alpha U(f, P)$.

So, αf is integrable. □

Theorem 22.7 (Thm 6.15 - Linearity)
Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ integrable. Then $f + g$ is integrable, and if α, β constants, then

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

Proof. In lecture notes. Proof uses $\inf_I \{f(x) + g(x)\} \geq \inf_I f(x) + \inf_I g(x)$ for interval I, and $\sup_I f(x) + \sup_I g(x) \geq \sup_I (f(x) + g(x))$ □

Example where $\inf_I \{f(x) + g(x)\} \neq \inf_I f(x) + \inf_I g(x)$:
Let $f(x) = x$ and $g(x) = 1 - x$ for $0 \leq x \leq 1$.

Then $\inf_{[0,1]} f(x) = 0 = \inf_{[0,1]} g(x)$ So $\inf f + \inf g = 0$.

$f(x) + g(x) = 1$, so $\inf(f(x) + g(x)) = 1$.