21 Integrals

Questions from chapter 4:

1. $g(x) = x^{3/2} \implies g'(0)$ exists? No, because g is not defined for $x < 0$.

   x needs a neighborhood in the domain to have a derivative at $x = 0$.

2. $g(x) = x + \sin x \implies$ g strictly increasing.
   $g'(x) = 1 + \cos x > 0$ if $x = \pi + 2n\pi$.
   The derivative can be equal to 0 at isolated points, and the function can still be strictly increasing.

3. $h(x) = x^{1/3} \implies h$ strictly increasing, h is continuous but $h'(0)$ DNE.

4. Let $f : [a, b] \to \mathbb{R}$, f continuous and differnetiable on (a, b). Must there be x_0 in (a, b) with $f'(x_0) = \frac{f(b)-f(a)}{b-a}$? No.

   $f(x) = 0$, $0 \le x < 1$, $f(1) = 1$

Note 21.1
Recall that $f : [a, b] \to \mathbb{R}$ is bounded, $P = \{a = x_0, x_1, \cdots, x_n = b\}$ a partition.

Then $L(f, P) = \text{lower sum} = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ where $m_i = \inf\{f(x); x_{i-1} \le x \le x_i\}$
$U(f, P) = \text{upper sum} = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ where $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$.

$L(f, P) \le U(f, P)$ always.

Definition 21.2
P^* is a **refinement** of P if P^* has all of the points of P.

Lemma 21.3 (Lemma 6.3)
If $f : [a, b] \to \mathbb{R}$ is bounded, P^* is a common refinement of partitions P and Q, then $L(f, P) \le L(f, P^*) \le U(f, P^*) \le U(f, Q)$. So every lower sum $L(f, P) \le$ every upper sum $L(f, Q)$.

Then, $\underline{\int}_a^b f = \sup_P L(f, P) \le \inf_P U(f, P) = \overline{\int}_a^b f$

Definition 21.4
If $\underline{\int}_a^b f = \overline{\int}_a^b f$, then f is **integrable** on $[a, b]$, and we write $\int_a^b f$, or $\int_a^b f(x)\,dx$.

Theorem 21.5 (Archimedes-Riemann Theorem - Thm 6.8 ***)
Let $f : [a, b] \to \mathbb{R}$ be bounded.
Then f is integrable on $[a, b]$ if and only if there is a sequence $\{P_n\}_{n=1}^{\infty}$ of partitions of $[a, b]$ with $\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$.

Proof. ($\implies$) Assume f is integrable, so $\sup_P L(f, P) = \inf_P U(f, P)$.
Then there are sequences of partitions $(P_n)_{n=1}^{\infty}, (Q_n)_{n=1}^{\infty}$ with $U(f, P_n) - L(f, Q_n) < \frac{1}{n}$ for $n \ge 1$.

Let P_n^* be a common refinement of P_n, Q_n.

Then $0 \le U(f, P_n^*) - L(f, P_n^*) \le U(f, P_n) - L(f, Q_n) \to 0$ (because $L(f, P_n) \le L(f, P_n^*) \le U(f, P_n^*) \le U(f, Q_n)$).

($\impliedby$) Assume $U(f, P_n) - L(f, P_n) \to 0$.
Then $\underline{\int}_a^b f = \overline{\int}_a^b f$, so f is integrable. ∎

Example 21.6 (Example 6.9 **)
Let $f : [a, b] \to \mathbb{R}$ be bounded and increasing. Then f is integrable.

Proof. Intermission. □

Definition 21.7
A partition of P of $[a, b]$ is **regular** if $P = \{a = x_0, x_1, \cdots, x_n = b\}$ and all of the subintervals have the same length $\frac{b-a}{n}$. gap P = length of the largest subinterval.

Proof. (Example 6.9) Let P_n be a regular partition of $[a, b]$ with gap $P_n = \frac{b-a}{n}$.

$U(f, P_n) = \sum_{i=1}^{n} f(x_i) \frac{b-a}{n}$ Let $\epsilon > 0$ be arbitrary. Then there is, by the Archimedean property, and n with $0 < \frac{b-a}{n}[f(b) - f(a)] < \epsilon$.

$U(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$ and $L(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(x_{i-1})$.

Then, $U(f, P_n) - L(f, P_n) = \frac{b-a}{n} \sum_{i=1}^{n}[f(x_i) - f(x_{i-1})] = \frac{b-a}{n}[f(b) - f(a)] < \epsilon$. So f is integrable. □

Example 21.8
Let $f(x) = x$, $0 \leq x \leq 1$. Show that f is integrable, and $\int_0^1 f(x)\,dx = \frac{1}{2}$

Solution: Since f is increasing on $[a, b]$, it is integrable by Example 6.9.

Let P_n be regular with gap $P_n = \frac{1}{n}$.
Then

$$U(f, P_n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right) \frac{1-0}{n} = \sum_{i=1}^{n} \frac{1}{n}\frac{1}{n} = \frac{1}{n^2} \sum_{i=1}^{n} i = \frac{1}{n^2} \frac{n(n+1)}{2} \to \frac{1}{2}$$

$$L(f, P_n) = \sum_{i=1}^{n} f\left(\frac{i-1}{n}\right) \left(\frac{1-0}{n}\right) = \sum_{i=1}^{n} \frac{i-1}{n} \frac{1}{n} = \frac{1}{n^2} \sum_{i=0}^{n-1} i = \frac{1}{n^2} \frac{(n-1)(n)}{2} \to \frac{1}{2}$$