# 20 L'Hopital's Rule, Partitions, Refinements, Upper/Lower Integrals

**Note 20.1**
Recall the Cauchy MVT: Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, differentiable on $(a, b)$. $g'(x) \neq 0$ for $x$ in $(a, b)$, and $g(a) \neq g(b)$. Then there is an $x_0$ in $(a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

**Theorem 20.2** (L'Hopital's Rule)
Assume $f(a) = 0 = g(a)$. Assume $g(x) \neq 0$, $g'(x) \neq 0$ for $a < x < b$. Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

*Proof.* By the Cauchy MVT, for an $x$ in $(a, b)$, there is $z_x$ in $(a, x)$ with

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(z_{ax})}{g'(z_{ax})}$$

Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{z_{ax} \to a^+} \frac{f'(x)}{g'(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

□

Note that L'Hopital's rule works for $x \to a^-$, $x \to a$, and also if $\lim_{x \to ?} f(x) = \pm\infty = \lim_{x \to ?} g(x)$

**Example 20.3**
Show $\lim_{x \to 0} \frac{\sin 2x}{3x} = \frac{2}{3}$.

Solution: $f(x) = \sin 2x$, $g(x) = 3x$, $a = 0$. Then

$$\lim_{x \to 0} \sin 2x = 0 = \lim_{x \to 0} 3x$$

By L'Hopital's rule,

$$\lim_{x \to 0} \frac{\sin 2x}{3x} = \lim_{x \to 0} \frac{2 \cos x}{3} = \frac{2}{3}$$

## 20.1 Partitions, Refinements, Integrals - Chapter 6

**Definition 20.4**
Let $a < b$ and $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.
Then $P = \{a = x_0, x_1, \cdots, x_n = b\}$ is a **partition** of $[a, b]$.

A partition is a finite set of points.

**Definition 20.5**
Let $f : [a, b] \to \mathbb{R}$ be bounded, and $P = \{a = x_0, x_1, \cdots, x_n = b\}$ is a partition of $[a, b]$.

Then the **lower sum** is

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

Where $m_i = \inf\{f(x) : x_{i-1} \le x \le x_i\}$.
The **upper sum** is

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

Where $M_i = \sup\{f(x) : x_{i-1} \le x \le x_i\}$.

Note that $L(f, P) \le U(f, P)$.

**Example 20.6**
Let $f(x) = \sin x$, $0 \le x \le 2\pi$, $P = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$.

$$L(f, P) = 0 \cdot \frac{\pi}{2} + 0 \cdot \frac{\pi}{2} - 1 \cdot \frac{\pi}{2} - 1 \cdot \frac{\pi}{2} = -\pi$$
$$U(f, P) = 1 \cdot \frac{\pi}{2} + 1 \cdot \frac{\pi}{2} + 0 \cdot \frac{\pi}{2} + 0 \cdot \frac{\pi}{2} = \pi$$

**Example 20.7**
$f(x) = x^2$, $-1 \le x \le 4$, $P = \{-1, 0, 2, 4\}$.

$$L(f, P) = 0 \cdot 1 + 0 \cdot 2 + 4 \cdot 2 = 8$$
$$U(f, P) = 1 \cdot 1 + 4 \cdot 2 + 16 \cdot 2 = 41$$

**Definition 20.8**
$P^*$ is a **refinement** of partition $P$ of $[a, b]$ if $P^*$ has all points of $P$, and usually additional points.

Also, $P^*$ is a **common refinement** of partitions $P, Q$ of $[a, b]$ if $P^*$ has all points of $P$ and of $Q$.

**Example 20.9**
$P = \{0, 1, 2, 4, 5\}$, $Q = \{0, \frac{1}{2}, 3, 4, 5\} \implies P^* = \{0, \frac{1}{2}, 1, 2, 3, 4, 5\}$ or $P^{**} = \{0, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1, 2, 3, 4, 5\}$
Both are common refinements.

**Lemma 20.10** (Lemma 6.3)
Let $P, Q$ be arbitrary partitions of $[a, b]$, and $f : [a, b] \to \mathbb{R}$ bounded.

Then $L(f, P) \le U(f, Q)$ (Every lower sum is less than or equal to every upper sum).

*Proof.* Let $P^*$ be a common refinement of $P, Q$.

$L(f, P) \le L(f, P^*)$, because when we add points, the infimums can only get larger.

$L(f, P) \le L(f, P^*) \le U(f, P^*) \le U(f, Q)$. (when we add points to the partitions, the supremums can only get smaller) □

**Definition 20.11**
Let $f : [a, b] \to \mathbb{R}$ be bounded.

Then the **lower integral** is

$$\underline{\int_a^b} f = \sup\{L(f, P) : P \text{ partition of } [a, b]$$

And the **upper integral** is

$$\overline{\int_a^b} f = \inf\{U(f, P) : P \text{ partition of } [a, b]$$

**Lemma 20.12** (Lemma 6.4)
Let $f : [a, b] \to \mathbb{R}$. Then $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

*Proof.* $\underline{\int_a^b} f = \sup_P L(f, P) \leq \inf_P U(f, P) = \overline{\int_a^b} f$ □

**Example 20.13**
Let $f(x) = c$, $a \leq x \leq b$, and $P = \{a = x_0, x_1, \cdots, x_n = b\}$.
Then $\sum_{i=1}^n m_i(x_i - x_{i-1}) = \sum_{i=1}^n c(x_i - x_{i-1}) = c(b - a)$

So, $L(f, P) = c(b - a) = \underline{\int_a^b} f$

Then $\sum_{i=1}^n M_i(x_i - x_{i-1}) = \sum_{i=1}^n c(x_i - x_{i-1}) = c(b - a)$

If $f(x) = c$, $a \leq x \leq b$, then $\underline{\int_a^b} c = c(b - a) = \overline{\int_a^b} c$

**Example 20.14**
Let $g(x) = \begin{cases} 0 & x \text{ is rational in } [0, 1] \\ 1 & x \text{ is irrational in } [0, 1] \end{cases}$, $P = \{a = x_0, x_1, \cdots, x_n = b\}$.
Then $\underline{\int_0^1} g = 0$ Since $m_i = 0$ for all $i$.
So $\overline{\int_0^1} g = 1$ Since $m_i = 1$ for all $i$.