2 Density, Absolute Value, Power Formula, Binomial Formula

2.1 Diagnostic test review

Question 1: P is equivalent to Q is equivalent to $P \implies Q$ and $Q \implies P$ ($P \iff Q$)

Question 2: To prove the following statement by contradiction: If $x + \frac{1}{x} < 2$, then $x < 0$,
First step: Assume $x \geq 0$

Question 3: Negation of: "For all x, there is y such that $xy = 1$":
The negation of this statement is "There is x such that for all y, $xy \neq 1$".

Question 4: Prove $n^2 \geq 2n - 1$ for all integers n.
Proof: $n^2 \geq 2n - 1 \iff n^2 - 2n + 1 \geq 0 \iff (n-1)^2 \geq 0$, which is true for all n.
Note that the above if and only ifs can be read as "is equivalent to"

2.2 Archimedean property continued

Example 2.1 (Problem 1.2.9)
Show that the Archimedean Property is a consequence of the fact that every open interval (a, b) has a rational number.

Proof. Assume $0 < a < b$, and let $c = a$.
Then there is a rational number $\frac{p}{q}$ such that $a < \frac{p}{q} < b$, with $p > 0, q > 0$ where $p, q \in \mathbb{Z}$.
Then

$$c \leq cq = aq < p$$

Then for any c, there is an integer p that is greater than it. So, these two properties (every open interval (a, b) has a rational number and the Archimedean principle) are equivalent. □

2.3 Density

Definition 2.2
A set S is dense in $\mathbb{R}$ if every non-empty open interval (a, b) has an element of S.

Theorem 2.3
The rationals $\mathbb{Q}$ are dense in $\mathbb{R}$.

Proof. by the theorem from lecture 1 (there is a rational in every open interval). □

Corollary 2.4
The set I of irrationals are dense in $\mathbb{R}$.

Proof. Let $a < b$ with a, b arbitrary. Then there are rationals r, s with $a < r < s < b$.
Let $t = r + \frac{1}{\sqrt{2}}(s - r)$. So t is irrational, and $a < t < r + (s - r) = s < b$. So the irrational t is in (a, b). □

2.4 Absolute values

Definition 2.5
For any real x, $|x|$ = absolute value of x is the larger of x and $-x$.
If $x \geq 0$, then $|x| = x$. If $x < 0$, then $|x| = -x$.

Note 2.6

1. $|x|$ is nonnegative
2. $-|x| \leq x \leq |x|$
3. $|x|$ = distance between x and 0
4. If $|x| \leq d$, then $-d \leq x \leq d$
5. $|b - a| < d \iff -d < b - a < d$, so that $a - d < b < a + d$, so b is in $(a - d, a + d)$

2.5 Triangle inequality

Theorem 2.7 (Triangle Inequality - Thm 1.11)

$$|a + b| \leq |a| + |b|, \forall a, b$$

Proof. $a \leq |a|$, $b \leq |b|$, so $a + b \leq |a| + |b|$
Also, $-(a + b) = -a - b \leq |a| + |b|$
Thus, $|a + b| \leq |a| + |b|$. □

2.6 Power formula

The power formula is as follows:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$$

We can see that the RHS of the equation reduces to

$$a^n + \cancelto{0}{a^{n-1}b - a^{n-1}b} + \cdots - b^n$$

Note 2.8
If $n > 1$ and n odd, then we can factor $a^n + b^n$. For example, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

If $n > 1$ and n even, then there is no factoring of $a^n + b^n$. For example, $a^2 + b^2$

Example 2.9 (Geometric sum)
Let $a = 1$, $b = r$

$$1 - r^n = (1 - r)(1 + r + r^2 + \cdots + r^{n-1}) = (1 - r)\sum_{k=0}^{n-1} r^k$$

Then,

$$\sum_{k=0}^{n-1} r^k = \frac{1 - r^n}{1 - r} \text{ if } r \neq 1$$

Definition 2.10 (Binomial Coefficient & Formula)
Let $0 \leq k \leq n$, with $k, n \in \mathbb{Z}$. Then the binomial coefficient is

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

And the binomial formula is

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n$$

Example 2.11 (Problem 1.3.21)
Show that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof.

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} = \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k+1)!}$$

$$= \frac{n!(k+n-k+1)}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)} = \binom{n+1}{k}$$

□