19 Concavity, Cauchy MVT

Example 19.1 (Problem 4.3.6)
$f(x) = x^4 + 2x^2 - 6x + 2$. Show f has exactly 2 solutions.

Solution: $f(0) = 0 > 0$, $f(1) = 1 + 2 - 6 + 2 < 0$, f is contiuous, and so by the IVT there is a solution.

$f'(x) = 4x^3 + 2x - 6$, $f''(x) = 12x^2 + 4 > 0$ for all x.
Then f' is strictly increasing on $(-\infty, \infty)$. So, there can only be two solutions.

Example 19.2 (Problem 4.3.15)
Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, $g(x) \neq 0$ for all x, and $g(x)f'(x) = f'(x)g(x)$ for all x.
Show $f = cg$ for some constant c.

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = \frac{0}{(g(x))^2} = 0 \qquad \text{for all } x$$

Then by the Identity Criterion, $\frac{f}{g}(x) = c \implies f(x) = c(g(x))$ for all x.

Example 19.3 (Problem 4.3.24)
Let $f : \mathbb{R} \to \mathbb{R}$, f' and f'' existing for all x. Also, $f(0) = 0$, $f'(x) \leq f(x)$ for all x. Must $f \equiv 0$ (identically equal)?

No. $f(x) = 1 - e^x$. Then $f(0) = 0$, $f'(x) = -e^x \leq 1 - e^x = f(x)$ for all x.

Note 19.4
Recall the second derivative test:

Let x_0 be in open interval I, $f : I \to \mathbb{R}$ with f', f'' existing on I. $f'(x_0) = 0$.

Then if $f''(x_0) < 0$, then $f(x_0)$ is a local maximum value.
If $f''(x_0) > 0$, then $f(x_0)$ is a local minimum value.

Definition 19.5
Let f be defined on open interval I.

1. If $f'' > 0$ on I, then the graph of f is **concave upward** on I.
2. If $f'' < 0$ on I, then the graph of f is **concave downward** on I.
3. Let x_0 be in I. Assume $f'(x_0)$ exists, $f''(x)$ exists possibly at x_0.
   If f'' changes sign at x_0, then $(x_0, f(x_0))$ is an **inflection point** of the graph of f.

Example 19.6
$g(x) = x^{4/3} \implies g'(x) = \frac{4}{3}x^{1/3}$, $g''(x) = \frac{4}{9}x^{-2/3}$.

There is an inflection point at $x = 0$.

Example 19.7
$f(x) = 3x^4 - 4x^3$. Find local extreme values, concavity and inflection points.

Solution: $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1) = 0$ if $x = 0$ or $x = 1$.
$f''(x) = 36x^2 - 24x = 12x(3x - 2) = 0$ if $x = 0$ or $x = \frac{2}{3}$.
$f''(0) = 0$, so second derivative test doesn't say anything. But $f''(1) > 0 \implies f(1)$ is a local minimum value.

The graph is concave up on $(-\infty, 0)$ and $(\frac{2}{3}, \infty)$ (because $f'' > 0$ on that interval) and the graph is concave down on $(0, \frac{2}{3})$.

The inflection points are $(0, 0)$, $(\frac{2}{3}, -\frac{16}{27})$.

$\lim_{x\to\pm\infty} f(x) = \infty$, so we can now graph our function.

Example 19.8
$h(x) = |x| \implies h'(x) = 1$ if $x > 0 \implies h''(x) = 0$ if $x > 0$
$h'(x) = -1$ if $x < 0$, $h''(x) = 0$ if $x < 0$.

In this graph, there is no concavity.

Theorem 19.9 (Cauchy MVT - Thm 4.23)
Let $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ with f, g continuous on $[a, b]$, differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b), and $g(a) \neq g(b)$.

Then there is x_0 in (a, b) with $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$.

This is an extension of the mean value theorem: MVT is when $g(x) = x$, x in $[a, b]$.

Proof. Let $h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x)$ for $a \leq x \leq b$.

One can show $h(a) = \frac{f(a)g(b)-f(b)g(a)}{g(b)-g(a)} = h(b)$, and h is continuous on $[a, b]$, and $h'(x)$ exists on (a, b).

Rolle's theorem implies that there is an x_0 so that $0 = h'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x_0)$
So, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$. □

Theorem 19.10 (Thm 4.24)
Let I be an open interval, $n > 0$ integer. Let $f : I \to \mathbb{R}$, with $f^{(n)}(x)$ existing for x in I.

If $f^{(k)}(x_0) = 0$ for $k = 0, 1, \cdots, n-1$ and x_0 in I, then for each x in I, there is a z_n between x and x_0 such that $f(x) = \frac{f^{(n)}(z_n)}{n!}(x - x_0)^n$.

Proof. (Uses Thm 4.23 n times)

Let $g(x) = (x - x_0)^n$. Then $\frac{f(x)}{(x-x_0)^n} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)}$.
For some z_1 between x and x_0, $\frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(z_1)}{n(z_1-x_0)^{n-1}} = \frac{f'(z_1)-f'(x_0)}{g'(z_1)-g'(x_0)}$
For some z_2 between z_1 and x_0, $\frac{f'(z_1)-f'(x_0)}{g'(z_1)-g'(x_0)} = \frac{f''(z_2)}{n(n-1)(z_2-x_0)^{n-2}} = \cdots = \frac{f^{(n)}(z_n)}{n(n-1)\cdots 2(1)(z_n-x_0)^0} = \frac{f^{(n)}(z_n)}{n!}$.
So, $f(x) = \frac{f^{(n)}(z_n)}{n!}(x - x_0)^n$ □