18 Identity Criterion, Monotonicity, Second Derivative Test

Note 18.1
Recall that the Mean Value Theorem states:
Let $f : [a, b] \to \mathbb{R}$ be continuous, and differentiable on (a, b). Then there is an x_0 in (a, b) with $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Lemma 18.2 (Lemma 4.19)
Let $f : [a, b] \to \mathbb{R}$ be continuous, $f'(x) = 0$ for $a < x < b$. Then f is a constant function.

Proposition 18.3 (Identity Criterion **)
Let I be an open interval, and $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ with f, g differentiable on I. Then $f' = g'$ on I if and only if there is a constant C with $f(x) = g(x) + C$ for x in I.

Proof. ($\Longleftarrow$) Trivial, since if $f(x) = g(x) + C$ for all x in I, then $f' = g'$.

($\Longrightarrow$) Assume $f' = g'$ on I. Let $h(x) = f(x) - g(x)$ for x in I.
Then $h'(x) = f'(x) - g'(x) = 0$ for x in I.

By the above lemma, there is a constant C with $h(x) = C$ for all x in $[a, b]$.
Then $C = h(x) = f(x) - g(x)$, so $f(x) = g(x) + C$ for all x in I. □

Example 18.4
$f'(x) = 4x^3 - 6x^2 + 2x - 3 \implies f(x) = x^4 - 2x^3 + x^2 - 3x + C$. (This is the generalized anti derivative)
If $f(1) = 2$, then $2 = f(1) = 1 - 2 + 1 - 3 + C \implies C = 5$.

Corollary 18.5 (Corollary 4.21)
If $f' > 0$ on open interval I, then f is strictly increasing on I.

Proof. Let x, z be in I, with $x < z$. Then by the MVT, there is some c in (x, z) with $\frac{f(z)-f(x)}{z-x} = f'(c) > 0$.
Then $z > x \implies f(z) > f(x)$, so f is strictly increasing on I. □

Note 18.6
If f is continuous on $[a, b]$, $f' > 0$ on (a, b), then f is strictly increasing on $[a, b]$.

If f is continuous on $[a, b]$, $f' > 0$ on (a, b) except $x_1, x_2, \cdots, x_n$ in (a, b), then f is strictly increasing on $[a, b]$.

Example 18.7
Let $f(x) = x^3$. Then f is strictly increasing on $(-\infty, \infty)$ since $f'(x) = 3x^2 > 0$ if $x \neq 0$.

Let $g(x) = x + \sin x$. Then $g'(x) = 1 + \cos x > 0$ if $x \neq \pi + 2n\pi$ for all integers n.

Definition 18.8
Let $f : D \to \mathbb{R}$. Then $x_0 \in D$ is a **local maximizer** if there is a neighborhood U of x_0 with $U \subseteq D$ and $f(x_0) \geq f(x)$ for all x in U.

Similarly, z_0 is a local minimizer if $f(x) \geq f(z_0)$ on a neighborhood of z_0.

Theorem 18.9 (Second Derivative Test - Thm 4.22)
Let x_0 be in open interval I, $f : I \to \mathbb{R}$. Assume f' and f'' exist on I, and $f'(x_0) = 0$.

If $f''(x_0) < 0$, then x_0 is a local maximizer.
If $f''(x_0) > 0$, then x_0 is a local minimizer.

Proof. Let $f''(x_0) < 0$. Then for $x \approx x_0$ (x near x_0), we have

$$\frac{f'(x) - f'(x_0)}{x - x_0} = \frac{f'(x)}{x - x_0} \approx f''(x_0) < 0 \implies \text{if } x < x_0, \text{ then } f'(x) > 0. \text{ And if } x > x_0, \text{ then } f'(x) < 0.$$

So $f''(x_0) < 0 \implies x_0$ is a local maximizer. □

Example 18.10
$f(x) = x^4 - 2x^2$, find local maximizers and local minimizers.

Solution: $f'(x) = 4x^3 - 4x = 4x(x^2 - 1) = 0$ if $x = -1, 0,$ or 1.

$f''(x) = 12x^2 - 4$ and $f''(0) < 0 \implies f(0) = 0$ is a local maximum value.
$f''(x) = 12x^2 - 4$ and $f''(\pm 1) > 0 \implies f(\pm 1) = -1$ is a local minimum value.