17 Rolle's Theorem, Mean Value Theorem

Note 17.1
Let $f : I \to \mathbb{R}$. Then x_0 in I is a maximizer if $f(x_0) \geq f(x)$ for x in I.
Also, x_0 in I is a minimizer if $f(x_0) \leq f(x)$ for x in I.

Lemma 17.2 (Lemma 4.16)
Let I be a neighborhood of x_0. Assume $f : I \to \mathbb{R}$ and $f'(x_0)$ exists.
If x_0 is a maximizer or minimizer, then $f'(x_0) = 0$.

Proof. By contradiction.
Assume $f'(x_0) > 0$. Then if $x \approx x_0$ (x is near x_0) and $x > x_0$, then $\frac{f(x)-f(x_0)}{x-x_0} \approx f'(x_0) > 0$, so $f(x) > f(x_0)$.
Then, x_0 is not a maximizer.

If $x \approx x_0$ and $x < x_0$, then $\frac{f(x)-f(x_0)}{x-x_0} \approx f'(x_0) > 0$, so $f(x) < f(x_0)$, so x_0 is not a minimizer. □

Theorem 17.3 (Rolle's Theorem)
Let $f : [a, b] \to \mathbb{R}$, with f continuous on $[a, b]$, f' exists on (a, b).

If $f(a) = f(b)$, then there is x_0 in (a, b) with $f'(x_0) = 0$.

Proof. Case 1: If f is constant, then $f'(x_0) = 0$ for all x_0 in (a, b).

Case 2: If f is not constant, assume there is x in (a, b) with $f(x) > f(a) = f(b)$.
By the Extreme Value Theorem, there is an x_0 in (a, b) with $f(x_0)$ being a maximum value.
By our above lemma, this means that $f'(x_0) = 0$. □

Theorem 17.4 (Mean Value Theorem - Thm 4.18 ***)
Assume $f : [a, b] \to \mathbb{R}$, f continuous, f' exists on (a, b). Then there is an x_0 in (a, b) with $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Proof. Equation for the line L connecting $f(a)$ and $f(b)$: $y = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$

Let $h(x) = f(x) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(x - a)\right]$ = distance from the graph of f to L.

h is continuous on $[a, b]$, h is differentiable on (a, b).
$h(a) = f(a) - f(a) = 0 = h(b)$. So by Rolle's theorem, there is an x_0 in (a, b) with $h'(x_0) = 0$.

$h'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a}$, so $f'(x_0) = \frac{f(b)-f(a)}{b-a}$. □

Example 17.5
$f(x) = x^3 + ax^2 + bx + c$, with a, b, c constants. Show f has ≤ 3 solutions.

Solution: $f'(x) = 3x^2 + 2ax + b$, so the maximum number of solutions to f' is 2.

By Rolle's theorem, between any 2 solutions of f is a solution of f'.
So, the max number of solutions of f is 3.

Example 17.6
$f(x) = x^3 + ax^2 + bx + c$ has ≥ 1 solutions.

Solution: Since $\lim_{x \to -\infty} f(x) = -\infty$, and $\lim_{x \to \infty} f(x) = \infty$, and f is continuous on $\mathbb{R}$, so we use the IVT.

Example 17.7
$f_1(x) = x^3$ has 1 solution.
$f_2(x) = x^2(x-1)$ has two solutions.
$f_3(x) = x(x-1)(x-2)$ has 3 solutions.

Lemma 17.8 (Lemma 4.19)
Let $f : [a, b] \to \mathbb{R}$ be continuous, $f'(x) = 0$ for $a < x < b$. Then f is constant.

Solution: Let $a < x < b$.
By the MVT, there is a z_x with $a < z_x < x$ and $\frac{f(x)-f(a)}{x-a} = f'(z_x) = 0$. So $f(x) = f(a)$.
Then $f(a) = f(x)$ for all x in (a, b). f continuous on $[a, b]$ means f is constant on $[a, b]$.

Proposition 17.9 (Identity Criterion - Prop 4.20 **)
Let I be an open interval, $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ differentiable on I.
Then $f' = g'$ on $I \iff$ there is a constant C with $f(x) = g(x) + C$ for all x in I.