16 Derivative Rules

Example 16.1 (Problem 4.1.14)
$f : \mathbb{R} \to \mathbb{R}$ is differentiable at x_0.
Then

$$\begin{aligned}\lim_{h\to 0} \frac{f(x_0+h)-f(x_0-h)}{h} &= \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h} + \frac{f(x_0)-f(x_0-h)}{h}\right)\\ &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} + \lim_{h\to 0}\frac{f(x_0-h)-f(x_0)}{-h}\\ &= f'(x_0)+f'(x_0) = 2f'(x_0)\end{aligned}$$

16.1 Differentiation Rules

Assume $f'(x_0)$ and $g'(x_0)$ exist.

1. Sum Rule:

$$\lim_{x\to x_0}\frac{(f+g)x-(f+g)x_0}{x-x_0} = \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}\right) = f'(x_0)+g'(x_0)$$

So, $\boxed{(f+g)'(x_0) = f'(x_0)+g'(x_0)}$

Note 16.2
Proposition 4.5 ** states that if $f'(x_0)$ exists, then f is continuous at x_0.

2. Product Rule

$$\begin{aligned}\lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)g(x)-f(x)g(x_0)}{x-x_0}+\frac{f(x)g(x_0)-f(x_0)g(x_0)}{x-x_0}\\ &= \lim_{x\to x_0} f(x)\left(\frac{g(x)-g(x_0)}{x-x_0}\right) + \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}g(x_0)\right]\\ &= \lim_{x\to x_0} f(x)\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} + \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\lim_{x\to x_0} g(x_0)\\ &= f(x_0)g'(x_0)+f'(x_0)g(x_0)\end{aligned}$$

So, $\boxed{(fg)'(x_0) = f(x_0)g'(x_0)+f'(x_0)g(x_0)}$

Note 16.3
If $g'(x_0)$ exists and $g'(x_0) \neq 0$, then

$$\left(\frac{1}{g}\right)'(x_0) = \lim_{x\to x_0}\frac{1/g(x)-1/g(x_0)}{x-x_0} = \lim_{x\to x_0}\left[\frac{g(x_0)-g(x)}{x-x_0}\frac{1}{g(x)g(x_0)}\right] = \frac{g'(x_0)}{(g(x_0))^2}$$

3. Quotient Rule: If $g(x) \neq 0$ for all x in the neighborhood of x_0, then

$$\left(\frac{f}{g}\right)' = \frac{g(x_0)f'(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}$$

For the proof, use $\frac{f}{g}(x) = (f(x)\frac{1}{g(x)})$ and use the Product Rule with the note above.

Note 16.4
All polynomials and all rational functions are differentiable.

Example 16.5
$f(x) = 3x^{15} - x^6 + \pi x - 4 \implies f'(x) = 45x^{14} - 6x^5 + \pi$

$g(x) = \frac{x^2-5}{x^3+1} \implies g'(x) = \frac{(x^3+1)(2x)-(x^2-5)(3x^2)}{(x^3+1)^2}$

Theorem 16.6 (Thm 3.29)
If I is an interval and $f : I \to \mathbb{R}$ is strictly monotone on I, then f^{-1} is continuous.

Theorem 16.7 (Thm 4.11)
Let I be a neighborhood of x_0, let $f : I \to \mathbb{R}$ where f is continuous and strictly monotone.

Assume $f'(x_0)$ exists, but $f'(x_0) \neq 0$. If $f(x_0) = y_0$, then $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.

Proof. Let $f(x_0) = y_0$, so $f^{-1}(y_0) = x_0$. Let $y_n \to y_0$.
Then $x_n = f^{-1}(y_n) \to x_0$ by Thm 3.29

Then
$$(f^{-1})'(y_0) \leftarrow \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \frac{x_n - x_0}{y_n - y_0} = \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{\frac{f(x_n)-f(x_0)}{x_n - x_0}} \to \frac{1}{f'(x_0)}$$

□

Example 16.8
$g(x) = x^n$ for $x \geq 0$ and $n \geq 1$. Show $(g^{-1})(x)$ exists for $x > 0$, and find a formula for it.

Solution: g is one to one, so g^{-1} exists and g^{-1} is strictly increasing.

Also, $g^{-1}(y) = x = y^{1/n}$ for $y \geq 0$.

Then
$$(g^{-1})'(y) = \frac{1}{g'(x)} = \frac{1}{nx^{n-1}} = \frac{1}{n(y^{1/n})^{n-1}} = \frac{1}{ny^{1-1/n}} = \frac{1}{n} y^{1/n-1}$$

Example 16.9
Let $f(x) = \sin x$ for $\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$. Find a formula for $(f^{-1})'(y)$ for $-1 < y < 1$.

Solution: $\sin x$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$, so f^{-1} exists on $[-1, 1]$.
Let y be in $(-1, 1)$.

Then
$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{\frac{d}{dx}\sin x} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\sqrt{1 - y^2}} \qquad \text{since } y = \sin x$$

Theorem 16.10 (Chain Rule - Thm 4.14)
Let $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ with $f(I) \subseteq J$. Assume $f'(x_0)$ exists, and $g'(f(x_0))$ exists.

Then $(g \circ f)'(x_0)$ exists, and $(g \circ f)'(x_0) = (g'(f(x_0)))\, f'(x_0)$

Proof. partial proof where $f(x) \neq f(x_0)$ for all x near x_0.

Let $f(x) = y$, $f(x_0) = y_0$. Then if $x_n \to x_0$, then

$$(g \circ f)'(x_0) \leftarrow \frac{g(f(x)) - g(f(x_0))}{x - x_0} = \frac{g(f(x)) - g(f(x_0))}{y - y_0} \left(\frac{y - y_0}{x - x_0} \right)$$
$$= \frac{g(y) - g(y_0)}{y - y_0} \frac{f(x) - f(x_0)}{x - x_0} \to (g'(f(x_0)))f'(x_0)$$

Example 16.11
$h(x) = (1 - x^2)^{3/2}$. Find $h'(x)$

Solution: Let $f(x) = 1 - x^2$ (inner function), and let $g(x) = x^{3/2}$ (outer function).

Then $h(x) = g(f(x))$, so by chain rule, $h'(x) = (g'(f(x))f'(x) = \frac{3}{2}(1 - x^2)^{1/2}(-2x)$

□