15 Derivatives

15.1 Derivatives

Definition 15.1
If I is an open interval and contains x_0, then the interval is a **neighborhood** of x_0.

Definition 15.2
Let f be defined on a neighborhood of x_0. Then f is **differentiable** at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists as a number. Then we write

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Note that

$$f'(x_0) = \frac{df}{dx}\bigg|_{x=x_0}$$

f is a **differentiable function** if $f'(x_0)$ exists for all x_0 in the domain.

Example 15.3
$f(x) = \sqrt{x}$ is not a differentiable function because $f'(0)$ does not exist.

$f(x) = c$, constant for all x, means that $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = \lim_{x \to x_0} \frac{c-c}{x-x_0} = 0$. So, $f'(x) = 0$ for all x.

$f(x) = x^n$ for all x, $n \in \mathbb{N}$ means that $f'(x_0) = nx_0^{n-1}$ for all $x_0 \in \mathbb{R}$.
Solution:

$$f'(x_0) = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x^{n-1} + x^{n-2}x_0 + \cdots + x_0^{n-1})}{x - x_0} = x_0^{n-1} + x_0^{n-1} + \cdots + x_0^{n-1} = nx_0^{n-1}$$

So, we have that $f(x) = c \implies f'(x) = 0$, and $g(x) = x^n \implies g'(x) = nx^{n-1}$ for all $n \in \mathbb{N}$

Note 15.4

$$h(x) = cx^n \implies h'(x_0) = \lim_{x \to x_0} \frac{cx^n - cx_0^n}{x - x_0} = c \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = cx_0^{n-1}$$

So, $h(x) = cx^n \implies h'(x) = ncx^{n-1}$

Example 15.5
$h(x) = \sqrt{x} \implies h'(x_0) = \frac{1}{2\sqrt{x_0}}$
Solution:

$$h'(x_0) = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{(\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \to x_0} \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \frac{1}{2\sqrt{x_0}}$$

$$f(x) = |x| \implies f'(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ \text{DNE} & x = 0 \end{cases}$$

Solution: $x_0 > 0$ and $x_0 < 0$ are obvious.
$x_0 = 0$: $\lim_{x \to 0} \frac{|x|-0}{x-0} = \lim_{x \to 0} \frac{|x|}{x}$ does not exist: $\lim_{x \to 0, x > 0} \frac{|x|}{x} = 1$, and $\lim_{x \to 0, x < 0} \frac{|x|}{x} = -1$

Note 15.6

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}\qquad\text{if } h=x-x_0$$

From this equivalent definition of the derivative, we can find that

$$\lim_{h\to 0}\frac{\sin h}{h}=1$$

And also that

$$\lim_{h\to 0}\frac{1-\cos h}{h}=\lim_{h\to 0}\left(\frac{1-\cos h}{h}\frac{1+\cos h}{1+\cos h}\right)=\lim_{h\to 0}\left(\frac{\sin^2 h}{h}\frac{1}{1+\cos h}\right)=\lim_{h\to 0}\left(\frac{\sin h}{h}\frac{\sin h}{1+\cos h}\right)=0$$

Example 15.7
$g(x)=\sin x \implies g$ is differentiable.
Solution: For x_0,

$$\begin{aligned}g'(x_0)&=\lim_{h\to 0}\frac{\sin(x_0+h)-\sin x_0}{h}\\&=\lim_{h\to 0}\frac{\sin x_0\cos h+\sin h\cos x_0-\sin x_0}{h}\\&=\lim_{h\to 0}\frac{(\sin x_0)(\cos h+1)}{h}+\lim_{h\to 0}\frac{\sin h}{h}\cos x_0\\&=\cos x_0\end{aligned}$$

Similarly, $g(x)=\cos h \implies g'(x)=-\sin x_0$.

Definition 15.8
Tangent line: Assume $f'(x_0)$ exists. Then the line L **tangent** to the graph of f at $(x_0, f(x_0))$ has the formula

$$\frac{y-f(x_0)}{x-x_0}=f'(x_0)$$

Example 15.9
$f(x)=\sqrt{x}$. Find the tangent line L tangent at $(4,2)$.
Solution:

$$f'(4)=\lim_{x\to 0}\frac{\sqrt{x}-\sqrt{4}}{x-4}$$

By our earlier example, we know that $f'(4)=\frac{1}{2\sqrt{4}}=\frac{1}{4}$.
Then L is given by

$$\frac{y-2}{x-4}=\frac{1}{4}\quad\text{or}\quad y=\frac{1}{4}(x-4)+2$$

Theorem 15.10 (Proposition 4.5 **)
If $f'(x_0)$ exists, then f is continuous at x_0.

Proof. Note $\lim_{x\to x_0} f(x)=f(x_0) \iff \lim_{x\to x_0}(f(x)-f(x_0))=0$.

$$\lim_{x\to x_0}(f(x)-f(x_0))=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}(x-x_0)=\left(\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right)\left(\lim_{x\to x_0}(x-x_0)\right)=0$$

So, f is continuous at x_0. □

Note that the converse is false. Example: $g(x)=|x|$