13 MATH410 Practice Exam 1 Spring 22

1. (a) Example: $a = \sqrt{2} = b$. a and b are irrational, but $ab = 2$.
   (b) False. $\{(-1)^n\}_{n=1}^{\infty}$ diverges, $a_n = (-1)^n$. But $a_{2n} = 1$ for $n \geq 1$, so $\{a_{2n}\}_{n=1}^{\infty}$ converges.
   (c) True. $\{n\}_{n=1}^{\infty}$ is discrete, so it is continuous.
   (d) True by the extreme value theorem.
2. (a) Law of induction: $S(n) = 1 + 3 + 5 \cdots + 2n - 1 > n^2 - 1$ for $n \geq 1$.

   Base case: $S(1) = 1 > 1^2 - 1 = 0$, OK.

   Induction hypothesis, assume $S(n)$ for an arbitrary $n \geq 1$.
   Then, for $n + 1$: $1 + 3 + \cdots + (2n - 1) + (2n + 1) > (n^2 - 1) + 2n + 1 = (n + 1)^2 - 1$ □
   (b) The negation is: There exists a real number x such that for all real numbers $y \neq 0$, we have $xy \neq y^2 - y$.
3. (a) Monotone Convergence Theorem: Let $\{a_n\}_{n=1}^{\infty}$ be monotone. Then $\{a_n\}_{n=1}^{\infty}$ converges if and only if it is bounded.
   (b) Prove every sequence has a monotone subsequence.

   Proof: Case 1: $\{a_n\}$ has an infinite collection $\{n_k\}_{k=1}^{\infty}$ of peak indices. Then $\{a_{n_k}\}_{k=1}^{\infty}$ is monotonically decreasing.
   Case 2: $\{a_n\}$ has only a finite number up to n^* of peak indices. Then $\{a_n\}_{n>n^*}^{\infty}$ must have a monotonically increasing subsequence.
4. (a) Let $f(x) = a_n x^n + \cdots + a_0$, where we assume $a_n > 0$ and n is odd.
   Then $\lim_{x\to\infty} f(x) = \infty$ since x^n dominates as $x \to \infty$.
   Also, $\lim_{x\to-\infty} f(x) = -\infty$ ...
   Let c be an arbitrary number. Then there are some a, b with $a < b$ and $f(a) < c < f(b)$.
   Because f is continuous on $[a, b]$, by the IVT, there is an x_0 in (a, b) with $f(x_0) = c$.

   So the range of f is $\mathbb{R}$.
   (b) $g(x) = \sqrt{x}$, $x \in [4, \infty)$. Prove g is uniformly continuous.

   Proof: Let $\{u_n\}, \{v_n\}$ be arbitrary in $[4, \infty)$ with $|u_n - v_n| \to 0$.
   Then $|g(u_n) - g(v_n)| = |\sqrt{u_n} - \sqrt{v_n}| = |\sqrt{u_n} - \sqrt{v_n}||\frac{\sqrt{u_n}+\sqrt{v_n}}{\sqrt{u_n}+\sqrt{v_n}}| = \frac{|u_n - v_n|}{\sqrt{u_n}+\sqrt{v_n}} \leq \frac{|u_n - v_n|}{\sqrt{4}+\sqrt{4}} \to 0$, so g uniformly continuous.
5. (a) $h(x) = x^2$. use epsilon elta to prove it is continuous at $x_0 = 4$.

   Proof: Let $\epsilon > 0$ be arbitrary. To find $\delta > 0$ so $|h(x) - h(4)| = |x^2 - 4^2| < \epsilon$ if $|x - 4| < \delta$.

   If $|x - 4| < 1$, then $3 < x < 5$. Let $\delta = \min(1, \epsilon/9)$.

   Then $|x^2 - 4^2| = |x - 4||x + 4| < \frac{\epsilon}{9} \cdot 9 = \epsilon$.
   (b) $\lim_{x\to0,x>0} \frac{1-2/x}{1-1/\sqrt{x}} = \lim_{x\to0,x>0} \frac{(x-2)1/x}{(\sqrt{x}-1)1/\sqrt{x}} = \lim_{x\to0,x>0} \left(\frac{x-2}{\sqrt{x}-1}\right)\left(\frac{1}{\sqrt{x}}\right) = \infty$.