12 MATH410 Practice Exam 1 Fall 22

1. (a) $f : [0,1] \to \mathbb{R}$. Let $f(x) = \pi$, $0 \le x \le 1$. Then all $0 \le x \le 1$ are maximizers.
   Also take $g : [0,1] \to \mathbb{R}$: Let $g(0) = 0$, $g(x) = \sin \frac{1}{x}$ for $0 \le x \le 1$.
   Then $g\left(\frac{1}{2n\pi+\pi/2}\right) = \sin(2n\pi + \pi/2) = 1$, for $n \ge 1$.

   (b) Take f from 1a: the function is monotone, but not strictly monotone.

   (c) Proof (by induction):

   Base case: $n = 1$. We have $2 = 1(2)$.

   Induction Hypothesis: Assume $2 + 4 + \cdots + 2n = n(n+1)$ for an arbitrary $n \ge 1$.
   Then $2 + 4 + \cdots + 2n + 2n + 2 = n(n+1) + 2n + 2$ by the IH.
   Then $n(n+1) + 2n + 2 = n^2 + 3n + 2 = (n+1)(n+2)$. □

2. (a) Assume that $\{c_n\}_{n=1}^{\infty}$ is bounded.

   Then if $\{c_n\}$ has an infinite number of peak indices $(n_k)_{k=1}^{\infty}$, then $(c_{n_k})_{k=1}^{\infty}$ is monotonically decreasing.
   If $\{c_n\}$ has only a finite number of peak indices, then there is a sequence $(n_k)_{k=1}^{\infty}$ such that $(c_{n_k})_{k=1}^{\infty}$ is monotonically increasing.

   Thus $\{c_n\}$ has a monotone subsequence.

   Then by the monotone convergence theorem, the subsequence converges because it is bounded and monotone.

   (b) $0 < a_n < 3$ for all $n \ge 1$, and $0 < r < 1$. $s_n = a_1 r + a_2 r^2 + \cdots + a_n r^n$. We wish to show this converges.
   Note: $0 \le s_n \le 3r + 3r^2 + \cdots 3r^n = 3r(1 + r + \cdots + r^{n-1}) \le \frac{3r}{1-r}$, so $\{s_n\}$ is bounded, and is monotonically increasing.
   Then by the monotone convergence theorem, $\{s_n\}$ converges.

3. (a) The extreme value theorem states that if $f : [a,b] \to \mathbb{R}$ is continuous, then f has a maximum value and a minimum value.

   (b) $g : [0,\infty) \to \mathbb{R}$. $g(x) = \frac{1}{\sqrt{x+1}} - 2x + x^3$ for $0 \le x$. Show that the range of g contains the interval $[1,\infty)$

   Solution: $g(0) = 1$. Also know $\lim_{x\to\infty} g(x) = \infty$.

   Let c be an arbitrary number greater than 1.
   Then $\lim_{x\to\infty} g(x) = \infty$ implies that there is some $b > 0$ with $g(b) > c$.

   Then $g(0) < c < g(b)$. Since g is continuous on $[0,\infty)$, so by the IVT, there is some x^* in $(0,b)$ with $g(x^*) = c$.

4. (a) $f(x) = \frac{2}{x}$ for $3 \le x \le 6$. Prove f is uniformly continuous.

   Solution: Let $\{u_n\}, \{v_n\} \subseteq [3,6]$ and arbitrary with $|u_n - v_n| \to 0$.

   Then, $|f(u_n) - f(v_n)| = |\frac{2}{u_n} - \frac{2}{v_n}| = 2\frac{|v_n - u_n|}{|u_n||v_n|} \le 2\frac{|v_n - u_n|}{3(3)} \to 0$. So f is uniformly continuous.

   (b) Let

   $$g(x) = \begin{cases} x - 1 & x < 0 \\ x + 1 & x \ge 0 \end{cases}$$

   Show g^{-1} exists and is continuous.

   Solution: $x < z \implies g(x) < g(z)$ if $x, z < 0$ or if $x, z > 0$, and if $x < 0, z > 0$, then $x - 1 < z + 1$. so g is one to one.

So, g^{-1} exists.

We have that
$x < 0 \implies y = g(x) = x - 1 \implies x = y + 1 \implies g^{-1}(x) = x + 1$ for $x < -1$.
$x \geq 0 \implies y = g(x) = x + 1 \implies x = y - 1 \implies g^{-1}(x) = x - 1$ for $x \geq 1$.

5. (a) $h(x) = x^2$. Use $\epsilon - \delta$ to prove h is continuous at $x = 3$.

Solution: Let $\epsilon > 0$ be aribtrary. To find $\delta > 0$ so that if $|x - 3| < \delta$, then $|h(x) - h(3)| < \epsilon$.

Then $|h(x) - h(3)| = |x^2 - 3^2| = |x - 3||x + 3|$. Let $\delta = \min(1, \epsilon/7)$. Then $|h(x) - h(3)| = |x - 3||x + 3| < \frac{\epsilon}{7}7 = \epsilon$.

(b) .

$$\lim_{x\to 0} \frac{1 - 2/x^2}{3 - 4/x} = \lim_{x\to 0} \frac{(x^2 - 2)1/x^2}{(3x - 4)1/x} = \lim_{x\to 0} \frac{x^2 - 2}{3x - 4}\frac{1}{x}$$

So, does not exist.