11 Limits

11.1 Limits

Note 11.1
Suppose f is uniformly continuous on set S, and $R \subseteq S$. Then f is uniformly continuous on T.

Proof. Let $\{u_n\}$, $\{v_n\} \subseteq T$ be arbitrary with $|u_n - v_n| \to 0$. Then $\{u_n\}$, $\{v_n\} \subseteq S$ and $|u_- v_n| \to 0$.
By theorem 3.17 (if f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$), $|f(u_n) - f(v_n)| \to 0$. $\square$

Example 11.2
Let $f(x) = x^5 + 3x^3 + 3$. Find an x_0 so that $f(x_0) = 4$.

Solution:
f is continuous on $\mathbb{R}$ since f is a polynomial. $f(0) = 3$, $f(1) = 7$, so by the IVT, there is an x_0 in $(0, 1)$ with $f(x_0) = 4$.

Question: is there a second $z_0 \neq x_0$ with $z_0 \in [0, \infty)$ and $f(z_0) = 4$?

Note: $0 < x$ and $0 < z$ with $x < z \implies f(x) < f(z)$.
Answer: No, there is no $z_0 > 0$, $z_0 \neq x_0$ and $f(z_0) = 4$.

Definition 11.3
If $D \subseteq R$, then x_0 is a **limit point** of D if there is a sequence $(x_n)_{n=1}^{\infty} \subseteq D$ with $x_n \neq x_0$ for all n, and $x_n \to x_0$.

Note 11.4
If $D = \mathbb{Q}$, then $\mathbb{R}$ is the collection of limit points of $\mathbb{Q}$.

Isolated points of D can not be limit points of D.
Thus, $\mathbb{N}$ has no limit points because they are all isolated.

$D = (0, 1] \implies$ the limits points are the set $[0, 1]$.

Definition 11.5
Let $f : D \to \mathbb{R}$, and x_0 is a limit point of D.
Then, $\lim_{x \to x_0} f(x) = L$ if whenever $(x_n)_{n=1}^{\infty} \subseteq D$, $x_n \to x_0$, $x_n \neq x_0$ for all n, then $f(x_n) \to L$.
We say L is the limit of $f(x)$ as $x \to x_0$.

Example 11.6

1. $\lim_{x\to 1} \frac{x^3-3x+2}{x-1} = \lim_{x\to 1} \frac{(x-1)(x-2)}{x-1} = \lim_{x\to 1} x-2 = -1$
2. $f(x) = \sqrt{x}$, $x \geq 0$. Then $\lim_{x\to 0} f(x) = \lim_{x\to 0, x>0} \sqrt{x} = 0$
3. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{\sqrt{x}-2}{(\sqrt{x}-2)(\sqrt{x}+2)} = \frac{1}{4}$
4. $g(t) = \sin t$. Show $\lim_{t\to 0} g(t) = 0$.

   Solution: Note that $0 \leq |g(t)| = |\sin t| \leq |t| \to 0$.
   So, $\lim_{t\to 0} g(t) = 0$.

5. $h(t) = \frac{1}{\sin t}$. Show that $\lim_{t\to 0} h(t)$ does not exist.

   Solution: Let $t_n = \frac{1}{n\pi + \pi/2}$. Then

   $$h(t) = \sin(n\pi + \frac{\pi}{2}) = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases}$$

   So, the limit does not exist.

Theorem 11.7 (Limit Rules)
Let $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$. Then

1. Sum Rule: $\lim_{x\to x_0} (f+g)x = L + M$
2. Product Rule: $\lim_{x\to x_0} (f(x)g(x)) = (\lim_{x\to x_0} f(x)) (\lim_{x\to x_0} g(x))$.

   In solution, $|f(x)g(x) - LM| \leq |f(x)g(x) - f(x)M| + |f(x)M - LM| = |f(x)||g(x) - M| + |f(x) - L|M \to 0$.
3. Quotient Rule: Note that we must have $M \neq 0$.

Theorem 11.8 (Composite Theorem - Thm 3.37)
$\lim_{x\to x_0} g(f(x)) = L$ if $\lim_{y\to y_0} g(y) = L$, etc.

Example 11.9
$\lim_{x\to 2} \sqrt{9-x^2} = \lim_{y\to 5} \sqrt{y} = \sqrt{5}$