10 Monotonicity, Inverse Functions

Note 10.1
A rational function is of the form $g(x) = \frac{p(x)}{q(x)}$ where p, q are polynomials. The function $f(x) = \sqrt{x}$ is not rational function.

Example 10.2
If f is uniformly continuous on $[a, b]$ and on $[b, c]$, then is f uniformlycontinuous on $[a, c]$?

Solution: Yes.
If f is uniformly continuous on $[a, b]$, then f is continuous on $[a, b]$.
If f is uniformly continuous on $[b, c]$, then f is continuous on $[b, c]$.

Then, f is continuous on $[a, c]$, and thus f is uniformly continuous on $[a, c]$.

10.1 Monotonicity

Definition 10.3
Let $f : D \to \mathbb{R}$. Then f is **monotone increasing** (i.e. increasing) if whenever x, z are in D, $x < z$, then $f(x) \leq f(z)$.

Let $f : D \to \mathbb{R}$. Then f is **monotone decreasing** (i.e. decrasing) if whenever x, z are in D, $x < z$, then $f(x) \geq f(z)$.

Note that both of these definitions allow for a constant function.

Let $f : D \to \mathbb{R}$. Then f is **strictly increasing** if whenever x, z are in D, $x < z$, then $f(x) < f(z)$.

Let $f : D \to \mathbb{R}$. Then f is **strictly decreasing** if whenever x, z are in D, $x < z$, then $f(x) > f(z)$.

Definition 10.4
$f : D \to \mathbb{R}$ is **monotone** if f is increasing or is decreasing.

Example 10.5
$f(x) = x^3$ is strictly increasing: $x < z \implies x^3 < z^3$

$g(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$ is increasing, not strictly increasing.

Theorem 10.6 (Thm 3.23)
Assume $f : D \to \mathbb{R}$ (read "f is a mapping from D to the reals") is monotone and $f(D)$ is an interval. Then f is continuous.

Proof. By contradiction. Assume f is increasing, and assume f is not continuous at x_0 in D.
Then there is an $\epsilon > 0$ such that there is a sequence $\{u_n\}_{n=1}^{\infty} \subseteq D$ with $u_n \to x_0$, but $|f(u_n) - f(x_0)| \geq \epsilon$ for all n.
In any case, $f(D)$ (the range) is not an interval (because there is an open space in there). Contradiction. □

10.2 Inverse Functions

Definition 10.7
$f : D \to \mathbb{R}$ is one-to-one (written 1-1) if whenever x, z are in D and $x \neq z$, then $f(x) \neq f(z)$.

Note 10.8
If f is 1-1, then it passes the horizontal line test (i.e. any horizontal line can touch the graph of f at most once).

If f is strictly monotone, then f is 1-1.
But note that the converse is not always true. Ex: $g(x) = \frac{1}{x}$ is 1-1, but is not monotone.

A function can be monotone but not 1-1. Ex: any constant function.

Definition 10.9
Let $f : D \to \mathbb{R}$ be 1-1. Then f has an inverse $f^{-1} : f^{-1}(y) = x \iff f(x) = y$.

Note 10.10
The domain of f^{-1} is the range of f, and vice versa.

The graph of f^{-1} is symmetric to the graph of f with respect to the line $y = x$.

$f^{-1}(f(x)) = x$ for all x in the domain of f.

f^{-1} is almost never equal to f. Exceptions: $f(x) = x$, and $g(x) = \frac{1}{x}$.

Example 10.11
$g(x) = x^3 + 2$. Show g^{-1} exists, and find a formula for it.

Solution:
$f(x) = x^3 \implies f$ is 1-1. Then $x \neq z \implies x^3 + 2 \neq z^3 + 2$, so g is 1-1.

To find g^{-1}: $y = x^3 + 2 \implies y - 2 = x^3 \implies (y-2)^{1/3} = x$. Then, $g^{-1}(x) = (x-2)^{1/3}$.

Note that here, we wrote g^{-1} as a function of x.