1 Introduction

1.1 Law of Induction ***

Used when we wish to prove a statement $S(n)$ is valid for all $n \geq n_0$ (usually $n_0 = 0$ or 1)

The steps are as follows:

1. Base case: $S(n_0)$ to prove
2. Induction Hypothesis: Assume $S(n)$ is valid for an arbitrary $n \geq n_0$. Then prove that $S(n+1)$ is valid.

Example 1.1
Prove that $2^n > n$ for $n \geq 1$.

Proof. By induction:
Base case: $2^1 = 2 > 1$, true.
Induction Hypothesis: Assume $2^n > n$ for an arbitrary $n \geq 1$.

$$2^{n+1} = 2(2^n) >_{IH} 2(n) = n + n \geq n + 1$$

Thus, the induction implies that $2^n > n$ for $n \geq 1$. □

1.2 Proof by Contradiction **

Used when we wish to prove that $P \implies Q$. We assume not Q, and prove not P.

Note
$P \implies Q$ is equivalent to not $Q \implies$ not P.

1.3 Rationals vs Irrationals

A real number x is rational if there are integers p, q with $x = \frac{p}{q}$, $q \neq 0$, and $\frac{p}{q}$ is in reduced form (no common nontrivial divisor of p and q)

$\frac{8}{6}$ is not in reduced form: $\frac{4}{3}$ is.

The irrational numbers are all non-rationals.

Example 1.2
Prove that $\sqrt{5}$ is irrational.

Proof. By contradiction: Assume $\sqrt{5} = \frac{p}{q}$ in reduced form, p, q integers.
Then, $p = \sqrt{5}q$, so $p^2 = 5q^2$, so $5|p^2$, so $5|p$ (by arithmetic laws)
Let $5r = p$, with $r \in \mathbb{Z}$. then $5q^2 = p^2 = 25r^2$, so $q^2 = 5r^2$, so $5|q$

Then, $5|p$ and $5|q$, so $\frac{p}{q}$ is not in reduced form. Contradiction. □

1.4 Upper bounds

Definition 1.3
A set $S \subseteq R$ has an **upper bound** b if $b \geq x$ for all x in S.
A set $S \subseteq R$ has a **lower bound** c if $c \leq x$ for all x in S.

Definition 1.4
If b is the smallest upper bound of S, then b is the **least upper bound** of S (lub S), which is the supremum of S ($\sup S$).
If c is the largest lower bound of S, then c is the **greatest lower bound** of S (glb S), which is the infimum of S ($\inf S$)

Axiom 1.5 (Completeness Axiom / Least Upper Bound Axiom)
If S has an upper bound, S has a $\sup S$.
If S has an lower bound, S has an $\inf S$.

Example 1.6
$S_1 = \{x \in \mathbb{R} : x^2 < 2\}$, $\sup S_1 = \sqrt{2}$, $\inf S_1 = -\sqrt{2}$
$S_2 =$ rationals < 1. $\sup S_2 = 1$, $\inf S$ does not exist. (The supremum and infimum must be numbers)
$S_3 =$ irrationals < 1. $\sup S_3 = 1$, $\inf S_3$ does not exist.
$S_4 = \{\frac{1}{n}\}_{n=1}^{\infty}$, $\sup S_4 = 1$, $\inf S_4 = 0$
$S_5 = \{1, 2, 3, \cdots\}$. $\sup S_5$ does not exist, $\inf S_5 = 1$

1.5 Archimedean Property ***

Property 1.7 (Archimedean Property)
For any $c > 0$, there is an integer $n > c$.
Equivalently, for any $c > 0$, there is an integer n with $0 < \frac{1}{n} < c$.

Theorem 1.8
Let $a < b$. Then there is a rational number in (a, b).

Proof. Assume $0 < a < b$. So, $b - a > 0$.
By the Archimedean Property, there is an integer n with $0 < \frac{1}{n} < \frac{b-a}{2}$.
Then, $\frac{2}{n} < b - a$, so $a + \frac{2}{n} < b$, so $a < a + \frac{1}{n} < a + \frac{2}{n} < b$.
Note that $(a + \frac{2}{n}) - (a + \frac{1}{n}) = \frac{1}{n}$, and so because there is a gap of length $\frac{1}{n}$, there must be some $\frac{k}{n}$ in $[a + \frac{1}{n}, a + \frac{2}{n}] \subseteq (a, b)$

If a $<$ b $<$ 0, then just look at $0 < -b < -a$, and find the rational in $(-b, -a)$

If $a < 0 < b$, then 0 is the rational. □