9 Introduction to Markov Chains (Section 3.1)

9.1 Introduction to Markov Chains (Section 3.1)

Definition 9.1
A **Markov chain** is a system relating states with a probability of moving (transitioning) between 2 states. The probability only depends on the current and next state, regardless of how long the process goes.

Examples of Markov chains include:

1. Insurance policies: states depend on age, health, marital status, etc.
Look at past statistics to determine what states a person might go to, and premiums might be raised in certain situations.
2. Mazes: which adjacent rooms might a player go to, with what probabilities?
If we move randomly through this maze, which room should I begin in to reach room 7 with the highest probability after 20 moves?
3. Cybersecurity graphs: states are vulnerable positions, some states cause a full breach. We wish to construct a system so that the probability of reaching a full breach is low.
4. Google PageRank: why does Wikipedia usually come up as the first result of a Google seearch? Because there are many websites that are linking to Wikipedia, making more popular.

Definition 9.2
The **transition matrix** P of a Markov chain with n states is an $n \times n$ matrix whose ijth entry is the probability of going from state j (column) to state i (row) in 1 transition.

Example 9.3
In a game, you win or lose. If you win, there is a 60% chance you win the next game. If you lose, there is a 25% chance you win next.

In our example, ROWS AND COLUMNS: W L

$$P = \begin{array}{c c} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{pmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{pmatrix} \end{array}$$

We assume a transtion always occurs (possibly to itself). Notice how the column sums must always be 1, because we always transition to a state (even if it is the one we are currently at).
These column sums are called **probability vectors** (non-negative values).

Square matrices whose columns are probability vectors are called **stochastic matrices**.

Definition 9.4
Let $\vec{x}_k$ be the **state vector** of a Markov chain: a column vector whose ith entry is the probability it is in state i after k transitions.

Example 9.5
In the game example, suppose you are given that you start in the W state.
Then, we have that

$$\vec{x}_0 = \begin{array}{c} W \\ L \end{array} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

And it follows that

$$\begin{array}{cc} & \begin{array}{cc} W & L \end{array} \\ \begin{array}{c} W \\ L \end{array} & \begin{pmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{pmatrix} \end{array} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \vec{x}_1$$

Which tells us that there is a 40% chance to go to the L state in 1 move, given that we start in W.

What if you start in W and want the probabilities after 2 transitions? We wish to find $\vec{x}_2$.

The two possibilities of state paths we can take to end up on a loss on the 2nd turn is $W \to L \to L$ or $W \to W \to L$.
We can calculate these probabilities manually: $(0.4)(0.75) + (0.6)(0.4)$. If we were to do calculations like this, our vector $\vec{x}_2$ would look like the following:

$$\vec{x}_2 = \begin{bmatrix} (0.6)(0.6) + (0.4)(0.25) \\ (0.6)(0.4) + (0.4)(0.75) \end{bmatrix}$$

But notice how we could do this more simply with our transition matrix:

$$\begin{bmatrix} 0.6 & 0.25 \\ 0.4 & 0.75 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = P\vec{x}_1 = P(P\vec{x}_0) = P^2\vec{x}_0 = \vec{x}_2$$

$\vec{x}_2$ is created using P^2! These state vectors depend on the powers of P!