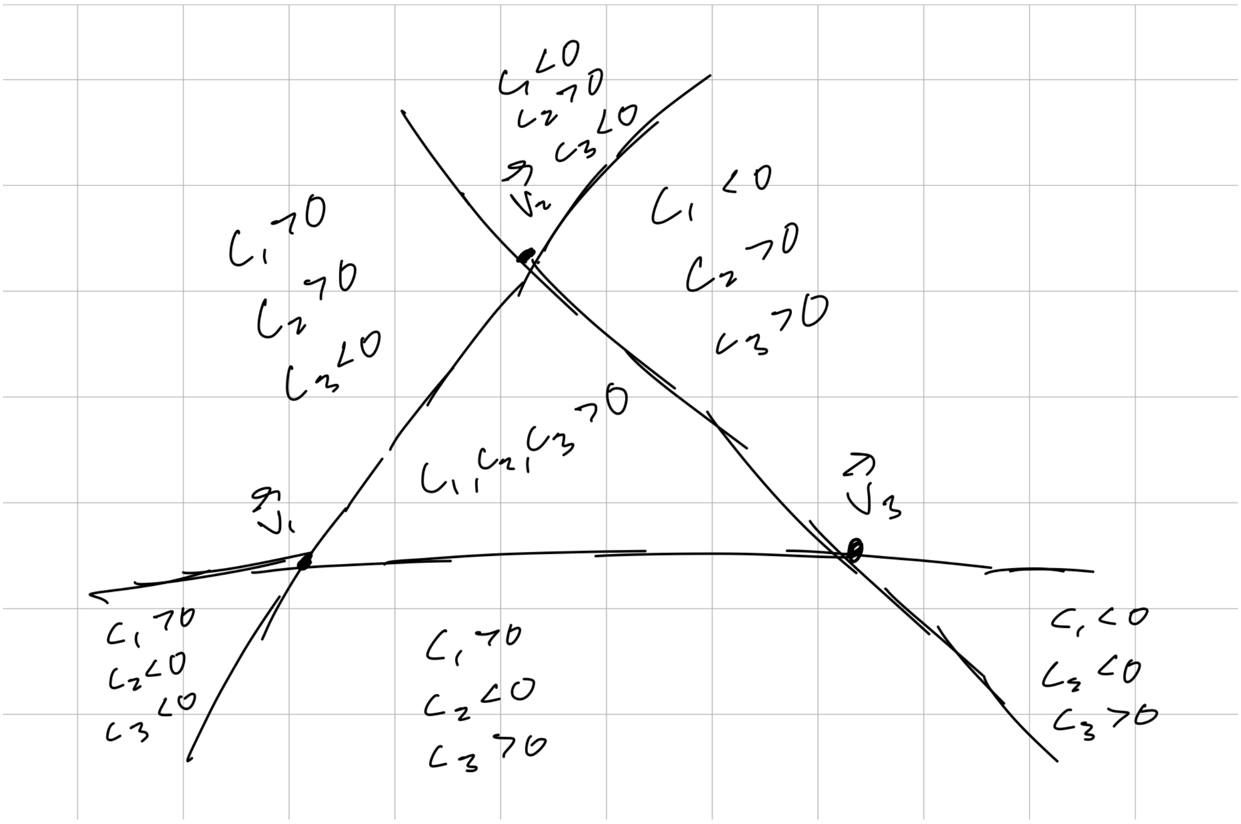
6 Convex Combinations, Singular Value Decomposition (Section 2.5)

6.1 Convex Combinations

Definition 6.1
A set S in $\mathbb{R}^n$ is **convex** if for all $\vec{p}, \vec{q} \in S$, the line segment $\overline{pq}$ is in S.

Equivalently, for all $\vec{p}, \vec{q} \in S$, $(1-t)p + tq \in S$, $0 \leq t \leq 1$.

Geometrically, a convex polygon required any line segment between two points of the polygon to stay "inside" of the polygon. This is a more formal definition of a convex polygon.



In $\mathbb{R}^2$, we divide the plane into 7 regions using the points of a triangle by extending out the sides of the triangle to be infinitely long lines.

Inside the triangle, is when all of the coefficients are positive (so they are inside of the convex hull).

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \qquad c_1 + c_2 + c_3 = 1$$

The points that lie on the side of the line formed by $\vec{v}_2$ and $\vec{v}_3$ that contains $\vec{v}_1$ has $c_1 > 0$. All regions that lie on the other side of the line have coefficient $c_1 < 0$. We can continue this logic to find the regions where c_2 and c_3 are positive or negative.

Example 6.2
Consider the triangle formed by $\{(-1,1),(1,4),(2,0)\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Use Barycentric coordinates to determine if $(0,2)$ and $(1, \frac{1}{4})$ lie in the triangle (Note this question is essentially asking to check if the point lies in the convex hull).

First we express $(0,2), (1, \frac{1}{4})$ as an affine combination by row reducing

$$\left[\begin{array}{ccc|cc} -1 & 1 & 2 & 0 & 1 \\ 1 & 4 & 0 & 2 & 1/4 \\ 1 & 1 & 1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 6/11 & 15/44 \\ 0 & 1 & 0 & 4/11 & -1/44 \\ 0 & 0 & 1 & 1/11 & 15/22 \end{array}\right]$$

And the right side of the row reduced augmented matrix represent the coefficients that form the affine combinations.
Thus, $(0,2)$ is in the triangle since all coefficients are ≥ 0, and $(1, \frac{1}{4})$ is not in the triangle since it is not a convex combination of the 3 points (since it has a negative coefficient).

Example 6.3
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
(Note that: $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$, $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u} \in \mathbb{R}^n, c \in \mathbb{R}$)

Let $S \subseteq \mathbb{R}^n$ be convex. Show that $M = \{T(\vec{u}) : \vec{u} \in S\}$ is convex.

We want to show that $(1-t)\vec{p} + t\vec{q} \in M$ for any $\vec{p}, \vec{q} \in M$, $0 \leq t \leq 1$.
Let $\vec{p}, \vec{q} \in M$. There exists $\vec{u}, \vec{v} \in S$ such that $T(\vec{u}) = \vec{p}, T(\vec{v}) = \vec{q}$.

This means that by the above linear transformation properties,

$$\begin{aligned}(1-t)\vec{p} + t\vec{q} &= (1-t)T(\vec{u}) + tT(\vec{v}) \\ &= T((1-t)\vec{u}) + T(t\vec{v}) = T((1-t)\vec{u} + t\vec{v})\end{aligned}$$

Since $\vec{u}, \vec{v} \in S$, we know that $(1-t)\vec{u} + t\vec{v} \in S$ because we assumed that S is convex.
Thus, we can conclude that $T((1-t)\vec{u} + t\vec{v}) \in M$, implying that $(1-t)\vec{p} + t\vec{q} \in M$, meaning that M is convex.

6.2 Singular Value Decomposition (SVD) (Section 2.5)

One of the most common uses of SVD is for image compression in computer science.

Theorem 6.4
Let A be $m \times n$. Then we can write

$$A = U\Sigma V^T$$

Here, U is an orthogonal $m \times m$ matrix (rows and columns are unit vectors, and all rows and columns are orthogonal, note that this matrix is not unique, we used gram-schmidt to find orthogonal matrices).
V is an orthogonal $n \times n$ matrix (not unique)
Σ is an $m \times n$ matrix whose upper left submatrix has positive entries that are non-increasing.