43 Final Exam Review

1. Let $P = \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 3/4 \end{bmatrix}$ be the transition matrix of a Markov chain. Determine what P^{1000} looks like (approximately).

   Solve $(P - I)\vec{q} = \vec{0}$ with $q_1 + q_2 = 1$.

$$\implies \vec{q} = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} \implies P^{1000} \approx \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

2. Consider the even cycle C_{2k}. Can equality be obtained in the Hoffman ratio bound?

   Note that because we can use the hoffman ratio bound, the graph must be regular. Here, because we have a cycle, C_{2k} is 2-regular.
   Recall

$$\alpha(G) \leq \frac{n}{1 - \frac{d}{\lambda_n}}$$

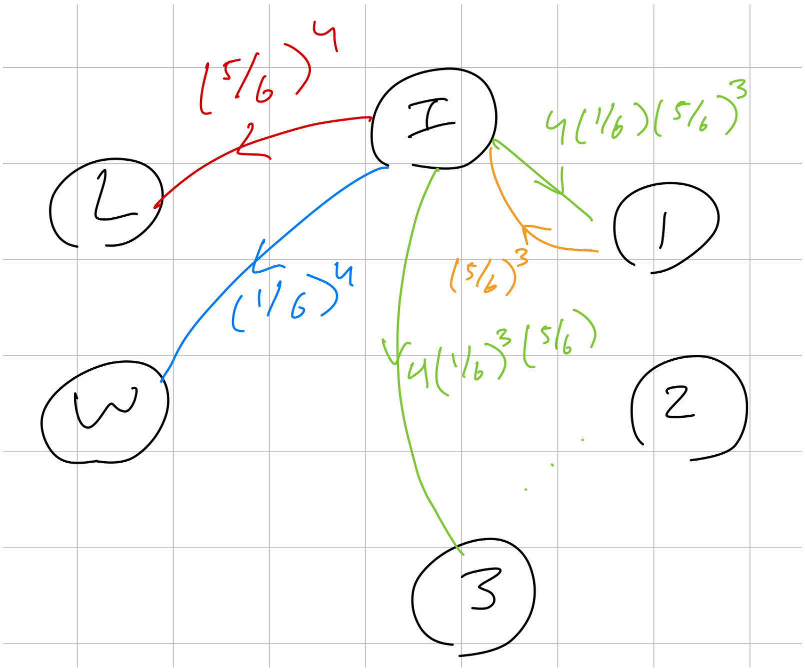
   C_{2k} having no odd cycles mean it is bipartite, so $\lambda = -2$ is the most neative eigenvalue.

$$\implies \alpha(C_{2k}) \leq \frac{2k}{1 - \frac{2}{-2}} = \frac{2k}{2} = k$$

   Equality can be obtained by picking every other vertex in the cycle.

3. You roll 4 dice. If any 6's occur, set them aside. If no 6's, you lose. If all 6's, you win. Otherwise, roll the remaining non-six dice.
   If no 6's, take one "saved" die back into play. Continue rolling until a win or loss.

   What are the expected number of plays?



   Then, with rows and columns corresponding to $W, L, I, 1, 2, 3$,

$$\implies P = \begin{bmatrix} 1 & 0 & (1/6)^4 & (1/6)^3 & (1/6)^2 & 1/6 \\ 0 & 1 & (5/6)^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & (5/6)^3 & 0 & 0 \\ 0 & 0 & 4(1/6)(5/6)^3 & 0 & (5/6)^2 & 0 \\ 0 & 0 & \binom{4}{2}(1/6)(5/6)^3 & 3(1/6)(5/6)^2 & 0 & 5/6 \\ 0 & 0 & \binom{4}{3}(1/6)(5/6)^3 & \binom{3}{2}(1/6)^2(5/6) & 2(1/6)(5/6) & 0 \end{bmatrix}$$

To answer the problem, we want Q, which is the 4×4 submatrix in the bottom right, corresponding to rows and columns $I, 1, 2, 3$.

$$(I - Q)^{-1} = \begin{bmatrix} 1.85 & & & \\ 1.46 & \vdots & \vdots & \vdots \\ 1.08 & & & \\ 0.43 & & & \end{bmatrix}$$

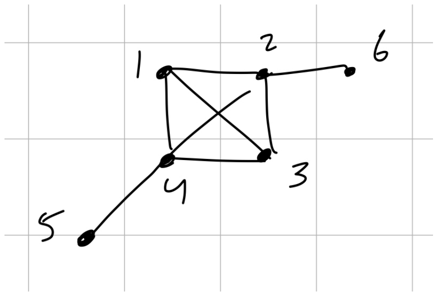
We only care about the I column because we start the game in I. The expected number of plays is $1.85 + 1.46 + 1.08 + 0.43 = 4.82$ plays.

With columns corresponding to $L, 1, 2, 3$, and rows corresponding to W, L:

$$S(I - Q)^{-1} = \begin{bmatrix} 0.11 & \vdots & \vdots \\ 0.89 & & \end{bmatrix}$$

So there is about an 11% chance of winning this game.

4. Take the following graph:



Which is the spectrum?

(a) $\{-3.7, -3.2, -2.1, (1)^{(2)}, 2\}$

(b) $\{-1.62, -1.39, -1, 0.23, 0.62, 3.16\}$

(c) $\{-3.22, (-1.5)^{(2)}, (1.5)^{(2)}, 3.22\}$

(d) $\{-1.5, -1.41, -1.3, -1.2, 2.31, 3.1\}$

$1 \to 3 \to 4 \to 1$ is an odd cycle, so the graph is not bipartite, so the spectrum can not be symmetric, so (c) is not the spectrum.

(a) can not be the specturm since the sum of the eigenvalues is clearly not 0.

Notice that we can decompose the graph into 3 bicliques, so $\text{bp}(G) \leq 3$. By the theorem in class, $bp(G) \geq \max\{\#\text{ pos}, \#\text{ neg evalues}\}$, which can be at most 3. This eliminates (d) since there are 4 negative eigenvalues.

So, the spectrum must be (b).

5. If A is the adjacency matrix, the **Laplacian matrix** is $L = D - A$, where D is the diagonal matrix whose ith diagonal entry is the degree of vertex i.
If $\vec{v}_i$ is an eigenvector for A with eigenvalue λ_i of a d-regular graph, find the spectrum for L.

Here, $L = D - A = dI - A \implies L\vec{v}_i = (dI - A)\vec{v}_i = dI\vec{v}_i - A\vec{v}_i = d\vec{v}_i - \lambda_i\vec{v}_i = (d - \lambda_i)\vec{v}_i \implies$ eigenvalues $d - \lambda_i$.