42 Name of Lecture

On final, no google matrix, polynomials, sets, SVD. Everything from second half will just be graph stuff. Average degree $\leq \lambda_1 \leq \Delta(G)$ can be a useful fact on the exam, but not strictly required.

1. A consumption matrix involves electricity, coal, and steel.
   With columns and rows corresponding to E, C, S.

$$\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.2 & 0.2 & 0.1 \\ 0.3 & 0 & 0.4 \end{bmatrix}$$

   (a) Describe the meaning of the value 0.5.
   You need 0.5 units of electricity to make 1 unit of coal.

   (b) If the open sector demands 87 of E, 87 of C, and 261 of S, how much total production is needed to satisfy all demands?

$$(I - A)\vec{x} = \vec{d}$$

   We want

$$\vec{x} = (I - A)^{-1} \begin{bmatrix} 87 \\ 87 \\ 261 \end{bmatrix} = \begin{bmatrix} 630 \\ 360 \\ 750 \end{bmatrix}$$

   So, we need 630 units of E, 360 units of C, and 750 units of S.

   One other thing to know about the input/output model:

$$(I - A)^{-1} = \begin{bmatrix} 100/87 & 100/87 & 370/261 \\ \cdots & & \\ \cdots & & \end{bmatrix}$$

   The value $\frac{370}{261}$ corresponds to the fact that if the open sector demands an additional 1 unit of steel, then we would need another $\frac{370}{261}$ units of electricity.

2. Let $\{(1, 1, 1), (2, 3, -2), (1, 3, 0), (4, -1, 4)\} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$

   (a) Find the Barycentric coordinates for $(7, -2, 2) = \vec{y}$ corresponding to this set.

   Recall to check if the set is affinely independent, check if $\{\vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_1, \vec{v}_4 - \vec{v}_1\}$ is linearly independent over $\mathbb{R}^3$.

   Asking for Barycentric coordinates implies that there must be a unique affine combination by definition.

   Here, we must write $\vec{y} - \vec{v}_1$ as a linear combination of $\{\vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_1, \vec{v}_4 - \vec{v}_1\}$.

$$\implies \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ 2 & 2 & -2 & -10 \\ -3 & -1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\implies 3(\vec{v}_2 - \vec{v}_1) - 7(\vec{v}_3 - \vec{v}_1) + 1(\vec{v}_4 - \vec{v}_1) = \vec{y} - \vec{v}_1$$

$$\implies 4\vec{v}_1 + 3\vec{v}_2 - 7\vec{v}_3 + \vec{v}_4 = \vec{y}$$

   So the Barycentric coordinates are $4, 3, -7, 1$.

   Recall that affinely dependency requires that $\sum c_i v_i = \vec{0}, \sum c_i = 0$ (where c_i not all 0)

3. Let $S = \{\vec{v} \in \mathbb{R} : \vec{u} \cdot \vec{z} = 5\}$ for some fixed vector $\vec{u} \neq \vec{0}$. Show that S is affine.

   We must show that for any $\vec{z}_1, \vec{z}_2 \in S$,
   $t\vec{z}_1 + (1 - t)\vec{z}_2 \in S$, $t \in R$.

Let $\vec{z}_1, \vec{z}_2 \in S$. Then we have

$$\vec{u} \cdot (tz_1 + (1-t)z_2) = \vec{u} \cdot (t\vec{z}_1) + (1-t)\vec{u} \cdot \vec{z}_2 = 5t + 5(1-t) = 5$$

$$\implies t\vec{z}_1 + (1-t)\vec{z}_2 \in S \implies S \text{ is affine}$$

From problem 2, what is the affine hull of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$?

We had $\{\vec{v}_2 - \vec{v}_1, \vec{v}_3 - \vec{v}_1, \vec{v}_4 - \vec{v}_1\}$ is a linearly independent set in $\mathbb{R}^3$. Thus this set forms a basis, meaning any vector $\vec{y} - \vec{v}_1$ is a linear combination of the set of vectors. Thus any $\vec{y} \in \mathbb{R}$ can be written as an affine combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.
So, the affine hull is $\mathbb{R}^3$.

4. The Sudoku graph of size n^2 (on n^4 vertices) is a graph whose vertices are the cells of a blank Sudoku puzzle, and two vertices are adjacent if and only if two cells are in the same row, column, or $n \times n$ block.

   (a) For the 9×9 graph ($n = 3$), what is the largest eigenvalue of the adjacency matrix?

   This graph is 20-regular, so the largest eigenvalue is $\lambda = 20$ with multiplicity 1.

   (b) Is the graph bipartite?

   No, the graph is not bipartite. We can create an odd cycle by traveling within 3 cells in the same row, column, or 3×3 cell.

   Coloring this graph requires all row, columns, and blocks to be different colors since they are all adjacent to each other.
   So, the sudoku puzzle can be seen as finding a proper 9-coloring of the graph, given a partial coloring of some vertices.