40 Graph Colorings (Section 5.6)

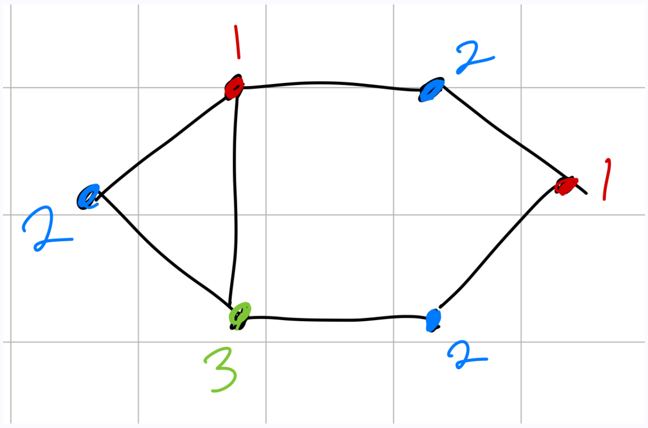
40.1 Graph Colorings (Section 5.6)

Definition 40.1
A **proper coloring** of a graph G is a labeling (coloring) of the vertices with 1 specific color so that adjacent vertices are different colors.

We say a graph is **k-colorable** if it can be properly colored with k colors.

The **chromatic number** of G, denoted $\chi(G)$, is the minimum k value so that G is k-colorable.

Example 40.2
An example of a coloring for a given graph:



Here, $\chi(G) = 3$.

Definition 40.3
Let A be a real, symmetric $n \times n$ matrix. The **Rayleigh Quotient** is

$$R(A, \vec{x}) = \frac{\vec{x}^T A \vec{x}}{\vec{x}^T \vec{x}} \qquad \vec{x} \neq \vec{0}$$

Theorem 40.4
Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a real, symmetric matrix. Then

$$\lambda_n \leq R(A, \vec{x}) \leq \lambda_1$$

For all $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$.

Suppose now A is the adjacency matrix, and we pick $\vec{x} = (1, 1, \cdots, 1)^T$ (G may NOT be regular).

$$R(A, \vec{x}) = \frac{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} [A] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} [A] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}{n} = \frac{\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \deg(v_1) \\ \deg(v_2) \\ \vdots \\ \deg(v_n) \end{bmatrix}}{n} = \frac{\sum_{i=1}^n \deg(v_i)}{n}$$

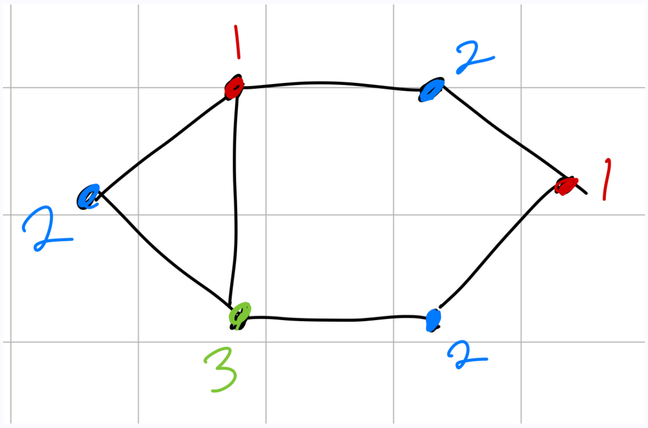
Which is the average degree of the graph.

Thus, we have the following:

Lemma 40.5
For any graph G, the largest eigenvalue λ_1 satisfies average degree $= R(A, \vec{x}) \leq \lambda_1 \leq \Delta(G)$.

Definition 40.6
The induced subgraph H of G is the graph where H is a subgraph of G, and for any vertices $u, v \in V(H)$, if $u \sim v$ in G, we must have $u \sim v$ in H.

Example 40.7
Take the following graph G:



The second graph is a subgraph of G, but it is not an induced subgraph because it is missing $1 \sim 3$. The third graph is an induced subgraph.

Theorem 40.8
If H is an induced subgraph of G, and $\lambda_k' \leq \cdots \lambda_1'$ are the eigenvalues of H, $\lambda_k \leq \cdots \leq \lambda_1$ are the eigenvalues of G, then $\lambda_1' \leq \lambda_1$ (largest eigenvalue is smaller)

Definition 40.9
We say G is **k-critical** if $\chi(G) = k$, but ANY proper subgraph H has $\chi(H) < k$.

Note that any odd cycle is 3 critical: any proper subgraph will be bipartite, so only 2 colors will be needed.

Theorem 40.10
If G is k-critical, then the minimum degree $= \delta(G) \geq k - 1 = \chi(G) - 1$

Now we know:

1. Average degree $\leq \lambda_1$
2. $\lambda_1(H) \leq \lambda_1(G)$ if H is an induced subgraph of G
3. If H' is k-critical, $\delta(H') \geq k - 1 = \chi(H') - 1 \implies \chi(H') \leq \delta(H') + 1$.

Theorem 40.11 (Wilf)
For any graph G with largest eigenvalue λ_1,

$$\chi(G) \leq 1 + \lambda_1$$

Proof. Given G, we can "reduce" G to an induced subgraph H' so that $\chi(H') = k$.
$\implies \chi(H') \leq_3 \delta(H') + 1 \leq \text{average degree} + 1 \leq_1 \lambda_1(H') + 1 \leq_2 \lambda_1(G) + 1.$

So, $\chi(G) \leq \lambda_1(G) + 1.$ □