4 Affine Combinations, Homogeneous Form and Graphics, Affine Independence

4.1 Affine Combinations Continued

Theorem 4.1
A vector $\vec{y} \in \mathbb{R}^n$ is an affine combination of $\vec{v}_1, \cdots, \vec{v}_m$ if and only if $\tilde{y} \in \text{span}(\{\tilde{v}_1, \cdots, \tilde{v}_m\})$, meaning the homogeneous form $\tilde{y}$ is a linear combination of the homogeneous forms $\tilde{v}_i$s.
Moreover, the coefficients of the linear combination ARE the coefficients of the affine combination.

Example 4.2
Last time, we had $\vec{y} = (17, 1, 5)$, $\vec{v}_1 = (-3, 1, 1)$, $\vec{v}_2 = (0, 4, -2)$, $\vec{v}_3 = (4, -2, 6)$. To find the coefficients of the affine combination, we row reduce the following matrix:

$$\left[\begin{array}{ccc|c} \tilde{v}_1 & \tilde{v}_2 & \tilde{v}_3 & \tilde{y} \\ \downarrow & \downarrow & \downarrow & \downarrow \end{array}\right] = \left[\begin{array}{ccc|c} -3 & 0 & 4 & 17 \\ 1 & 4 & -2 & 1 \\ 1 & -2 & 6 & 5 \\ 1 & 1 & 1 & 1 \end{array}\right]$$

Here, we can see that the last column represents $c_1 + c_2 + c_3 = 1$, which is solving the affine combination in disguise. So, this augmented matrix allows us to find coefficients that fulfill the condition for a linear combination to be affine.
After row reducing the matrix, we get:

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Which gives us the coefficients of the affine combination.

4.2 Homogeneous Form and Graphics (Section 2.2)

Given a point (x, y), we transform the object using various methods: rotating, expanding, and translating.
An inefficient way to denote the transformations applied on a point can be as follows: $(2((3, 4) + (1, 0))) + (0, 1)$ to describe a point $(3, 4)$ that is translated horizontally 1 unit, scaled by a factor of 2, then translated vertically 1 unit.

Recall that given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the **standard matrix** is

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ \downarrow & \downarrow & \downarrow & \downarrow \end{bmatrix}, \vec{e}_i = (0, \cdots, 0, \overbrace{1}^{i\text{th entry}}, 0, \cdots, 0)$$

Then the counterclockwise rotation matrix about the origin of θ radians is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If we look at transformations in $\mathbb{R}^2$ using homogeneous coordinates, then the rotation matrix is now

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

These two computations basically perform the same action, except the bottom computation leaves an extra coordinate of 1 at the bottom.

Now consider a translation/shift by k units on x, as well as m units on y. In homogeneous form, the matrix representation is for this translation is

$$\begin{bmatrix} 1 & 0 & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+k \\ y+m \\ 1 \end{bmatrix}$$

We were able to perform a translation ("adding") using matrix multiplication.
Now, every transformation can be represented by a simple product of matrices.

Example 4.3
Given point $p = (1, 2)$, determine the transformed point by first rotating counterclockwise about $(5, 4)$ at $\frac{\pi}{4}$ radians, and then scale the x-coordinate by $\frac{1}{2}$, and then swap x and y (reflect along $y = x$).

First, we shift the point so that $(5, 4)$ becomes the new "origin". Then, we can use the counterclockwise matrix to rotate our point, and then shift the point back to where it should be.

The operations performed on the vector, read from right to left, is a shifting left 5 units and down 4 units, a rotation of $\frac{\pi}{4}$ radians counterclockwise, a shifting right 5 units and up 4 units, a scaling of the x-coordinates by $\frac{1}{2}$, and then a reflection along $y = x$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \approx \begin{bmatrix} -0.2426 \\ 1.7929 \\ 1 \end{bmatrix}$$

Note 4.4
This is applicable to extend into $\mathbb{R}^3$, which would be useful for computer graphics applications.

4.3 Affine Independence (Section 2.3)

Definition 4.5
A set $\{\vec{v}_1, \cdots, \vec{v}_m\} \in \mathbb{R}^n$ is **affinely dependent** if there exists $c_1, \cdots, c_m$ NOT all zero such that

1. $\sum_{i=1}^{m} c_i \vec{v}_i = \vec{0}$ (linear dependence)
2. $\sum_{i=1}^{m} c_i = 0$

Otherwise, we say that the set is **affinely independent**.