39 Exam 3 Review

From last time, we had the Kneser graph, which has most negative eigenvalue $\mu = -\binom{n-r-1}{r-1}$, and is $\binom{n-r}{r}$-regular. So, the Hoffman ratio bound yields

$$\alpha(G) \le \frac{\binom{n}{r}}{1 - \frac{\binom{n-r}{r}}{-\binom{n-r-1}{r-1}}} = \binom{n-1}{r-1}$$

We have proven the following:

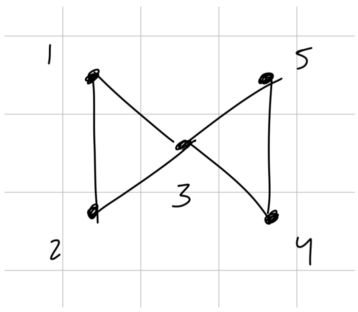
Theorem 39.1 (Erdos-Ko-Rado)
If $A_1, \cdots, A_m \subseteq [n]$ such that

1. $|A_i| = r$, $1 \le i \le m$
2. $|A_i \cap A_j| \ne 0$, $1 \le i \ne j \le m$

Then

$$m \le \binom{n-1}{r-1}$$

1. Take the following graph:



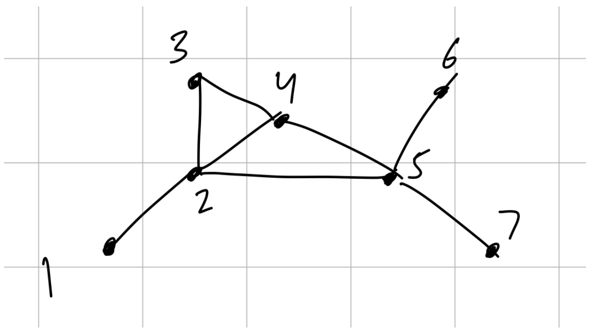
Let A be the adjacency matrix. Find $\mathrm{tr}(A^3)$.

This is essentially asking for the number of walks of length 3 that start and end at the same vertex.

Observe any C_3 can be formed starting at any of the 3 vertices, and we can walk clockwise or counterclockwise. So, each 3-cycle gets counted 3 times.
Then $\mathrm{tr}(A^3)$ is all the walks of length 3 with start and end nodes the same, which is 6· number of distinct triangles, which is 12.

More generally, the distinct number of triangles in any graph is $\frac{\mathrm{tr}(A^3)}{6}$. The only way that we can have a walk of length 3 that returns to the same node is by the existence of a 3-cycle.

2. Take the following graph:



(a) $\{-2.07, -1.37, -0.76, 0, 0.31, 1.06, 2.83\}$

(b) $\{-5, -0.79, (0)^{(2)}, 1.58, 1, 3.21\}$

(c) $\{-2.73, (-1.22)^{(2)}, 0, (1.22)^{(2)}, 2.73\}$

(d) $\{-3.81, (0.6)^{(5)}, 0.81\}$

The graph is not bipartite (explicit odd cycle: $3 \to 2 \to 4 \to 3$), so the spectrum can not be symmetric, so (c) is not the spectrum.

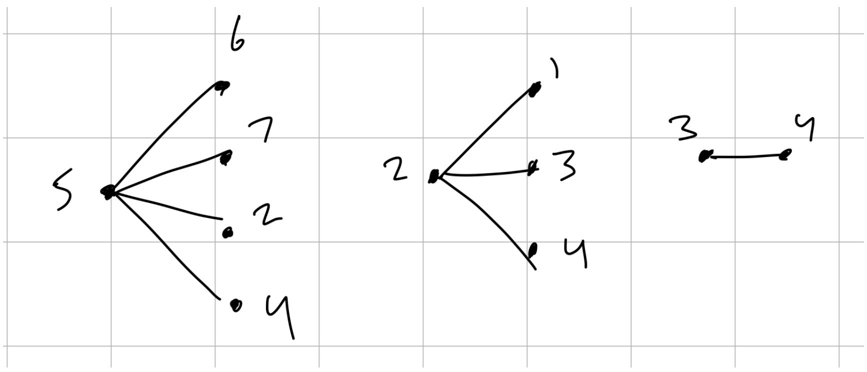
The maximum degree of the graph $\Delta(G) = 4 \implies |\lambda| \leq 4$, but (b) has eigenvalue -5, so (b) can not be the spectrum.

The diameter of this graph is 3 (largest distance between any two vertices), so there must be at least 4 distinct eigenvalues, so (d) can not be the spectrum.

Thus (a) is the spectrum.

What is $\mathrm{bp}(G)$? By the spectrum (a), we have that $\mathrm{bp}(G) \geq \max(\{\#\text{ pos}, \#\text{ neg evalues}\}) = 3$

We can decompose the graph into bicliques explicitly:



Thus from our explicit example, $\mathrm{bp}(G) \leq 3$. Combining, we find that $\mathrm{bp}(G) = 3$.

3. Let $A_1, \cdots, A_m \subseteq [n]$ such that $|A_i \cap A_j|$ is even, $|A_i|$ is odd.
Let $\vec{v}_1, \cdots, \vec{v}_m \in \mathbb{R}^n$ be the characteristic vectors. From the homework, we had the lemmas

(a) A is $m \times n$, $\mathrm{rank}(A) \leq \min(m, n)$

(b) $\mathrm{rank}(AB) \leq \min(\{\mathrm{rank}(A), \mathrm{rank}(B)\})$

Let M be the matrix whose ith row is $\vec{v}_i$.

(a) dimensions of MM^T?

M is $m \times n$, M^T is $n \times m$, so MM^T is $m \times m$.

(b) Describe the parity (even, odd) of the entries in MM^T.

$$\begin{bmatrix} \vec{v}_1 & \rightarrow \\ \vec{v}_2 & \rightarrow \\ \vdots & \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots \\ \downarrow & \downarrow & \end{bmatrix} = \begin{bmatrix} \text{odd} & & & \text{EVEN} \\ & \text{odd} & & \\ & & \ddots & \\ \text{EVEN} & & & \text{odd} \end{bmatrix}$$

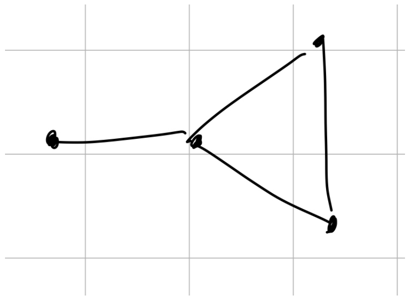
(c) Apply the cofactor expansion on row 1 of MM^T to explain why $\det(MM^T) \neq 0$.

$$\begin{bmatrix} \text{odd} & \text{even} & \text{even} & \text{even} \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Any entry that is even clearly expands out to create an even number. If we expand out on the odd number, the only contribution of an odd number comes from multiplying at the numbers on the diagonal.
$\implies \det(MM^T) = \text{even} + \text{even} + \cdots + \text{odd} \neq 0$

(d) What is the rank of MM^T?
$\det(MM^T) \neq 0 \implies$ invertible $\implies \text{rank}(MM^T) = m$
From here, we apply lemmas to conclude $m \leq n$.

(e) Take the graph



Which has spectrum $\{-1.48, 0.31, 2.17, x, y\}$. Find x and y.

We can easily see that $\text{bp}(G) = 2$.
Then, we can not have more than 2 positive eigenvalues. We also can not have 3 negative eigenvalues. We have $\text{tr}(A) = 0 \implies x + y = -1$. We know one of the eigenvalues must be 0, so the other one must be -1.