38 Hoffman Ratio Bound, Kneser Graph

Lemma 38.1
If G is d-regular, $d = \lambda_1 \geq \lambda_2, \cdots \geq \lambda_n$, and $\vec{x}_s = \sum_{i=1}^{n} a_i \vec{v}_i$, then $a_1 = \vec{v}_1 \cdot \vec{x}_s = \frac{|S|}{\sqrt{n}}$.

Proof.

$$\vec{v}_1 \cdot \vec{x}_S = \vec{v}_1 \cdot \left(\sum_{i=1}^{n} a_i \vec{v}_i \right) = a_1(\vec{v}_1 \cdot \vec{v}_1) = a_1(1) = a_1$$

Morover for $\lambda_1 = d$, we know it corresponds to eigenvector $(1, 1, \cdots, 1)^T$.
Normalized, we get that $\vec{v}_1 = \frac{1}{\sqrt{n}}(1, 1, \cdots, 1)^T$. So,

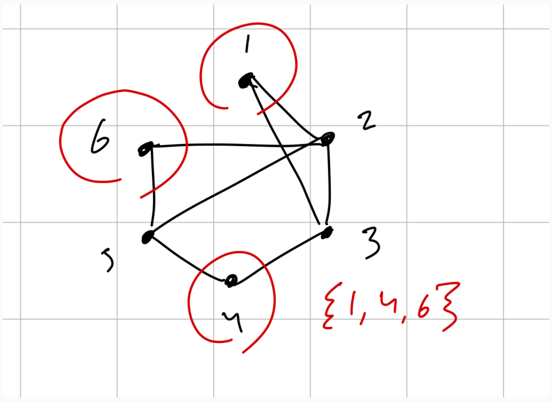
$$a_1 = \vec{v}_1 \cdot \vec{x}_S = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \vec{x}_S = \frac{1}{\sqrt{n}}|S|$$

□

Lemma 38.2
Let $\vec{x}_S$ be the characteristic vector as before.
Then

$$\vec{x}_S^T A \vec{x}_S = 0$$

Example 38.3
Take the following graph and independent set:



$$\implies A\vec{x}_S = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

Here, note that row 2, or y_2, is the number of vertices in the independent set that vertex 2 is adjacent to.

Proof.

$$\vec{x}_S^T A \vec{x}_S = \vec{x}_T \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = 0$$

If $x_i = 1$, then i is in the independent set. Then $y_i = 0$ since i can't be adjacent to anything in S. Then, $x_i y_i = 0$.
Else if $x_i = 0$, then $x_i y_i = 0$. □

Theorem 38.4 (Hoffman ratio bound)
If G is a connected d-regular graph with eigenvalues $d = \lambda_1 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq \frac{n}{1 - \frac{d}{\lambda_n}}$$

Example 38.5
Take the petersen graph, which has eigenvalues $\{3, (1)^{(5)}, (-2)^{(4)}\}$.
By the Hoffman bound,

$$\alpha(G) \leq \frac{10}{1 - \frac{3}{-2}} = 4$$

Equality can be obtained by construction.

Proof. Let $\vec{v}_1, \cdots, \vec{v}_n$ be an orthonormal basis of eigenvectors, and let $\vec{x}_S = \sum_{i=1}^{n} a_i \vec{v}_i$.
By lemma 3,

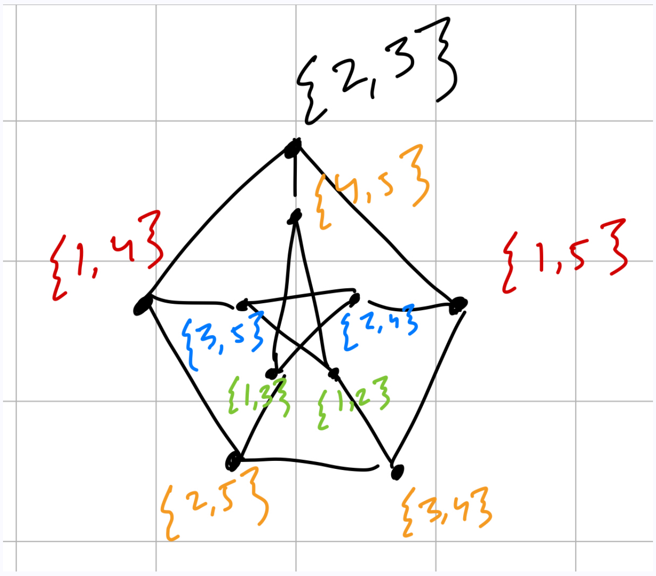
$$\begin{aligned}
0 =& \vec{x}_S \cdot (A\vec{x}_S) \\
&= \left(\sum_{i=1}^{n} a_i \vec{v}_i\right) \cdot \left(\sum_{i=1}^{n} a_i A\vec{v}_i\right) \\
&= \left(\sum_{i=1}^{n} a_i \vec{v}_i\right) \cdot \left(\sum_{i=1}^{n} a_i \lambda_i \vec{v}_i\right) \\
&= \sum_{i=1}^{n} \lambda_i a_i^2 \\
&= a_1^2 \lambda_1 + \sum_{i=2}^{n} \lambda_i a_i^2 \\
&= \left(\frac{|S|}{\sqrt{n}}\right)^2 (d) + \sum_{i=2}^{n} \lambda_i a_i^2 \qquad \text{by lemma 2} \\
&\geq \frac{d}{n}|S|^2 + \sum_{i=2}^{n} \lambda_n a_i^2 \\
&= \frac{d}{n}|S|^2 + \lambda_n(|S| - a_1^2) \qquad \text{by lemma 1} \\
&= \frac{d}{n}|S|^2 + \lambda_n\left(|S| - \frac{|S|^2}{n}\right) \qquad \text{by lemma 2} \\
\implies 0 &\geq \frac{d}{n}|S|^2 + \lambda_n|S| - \lambda_n\frac{|S|^2}{n} \\
\implies \frac{\lambda_n}{n}|S| &\geq \frac{d}{n}|S| + \lambda_n \\
|S|\left(\frac{\lambda_n}{n} - \frac{d}{n}\right) &\geq \lambda_n \\
|S|(\lambda_n - d) &\geq n\lambda_n \\
|S| &\leq \frac{n\lambda_n}{\lambda_n - d} = \frac{n}{1 - \frac{d}{\lambda_n}}
\end{aligned}$$

Because $\lambda_n - d$ is negative.

□

Definition 38.6
The **kneser graph** $K(n,r)$ is a graph whose vertices are the subsets of size r of $[n] = \{1, 2, \cdots, n\}$, with two vertices (sets) adjacent i f and only if the orresponding sets have no elements in common.

Example 38.7
$K(5,2)$ looks like the following:



$K(5,2)$ is the Petersen graph.

In general, $K(n,r)$ is on $\binom{n}{r}$ vertices, and is $\binom{n-r}{r}$-regular.

Theorem 38.8
The distinct eigenvalues of $K(n,r)$ are

$$\mu_i = (-1)^i \binom{n-r-i}{r-i}$$

This means that the most negative eigenvalue of $K(n,r)$ is

$$-\binom{n-r-1}{r-1}$$

i.e. the eigenvalue that occurs when $i = 1$.