37 Biclique Decomposition, Independence (Section 5.5)

Proof. (Continued) Assume m bicliques decompose G. Show $\text{Eig}_+(A) \cap U^\perp = \{\vec{0}\}$.
Let $\lambda_1, \cdots, \lambda_k$ be the positive eigenvalues.

For any $\vec{y} \neq \vec{0}$ in $\text{Eig}_+(A)$, write

$$\vec{y} = \sum_{i=1}^{k} c_i \vec{w}_i \implies A\vec{y} = \sum_{i=1}^{k} c_i A(\vec{w}_i) = \sum_{i=1}^{k} c_i \lambda_i \vec{w}_i$$

$$\implies \vec{y}^T A \vec{y} = \vec{y} \cdot (A\vec{y}) = \sum_{i=1}^{k} c_i \vec{w}_i \cdot \left(\sum_{i=1}^{k} c_i \lambda_i \vec{w}_i \right)$$

Recall that we have an orthonormal basis of eigenvectors, so $\vec{w}_i \cdot \vec{w}_j = 0$ when $i \neq j$, and $\vec{w}_i \cdot \vec{w}_i = ||\vec{w}_i||^2 = 1$. So,

$$\sum_{i=1}^{k} c_i \vec{w}_i \cdot \left(\sum_{i=1}^{k} c_i \lambda_i \vec{w}_i \right) = \sum_{j=1}^{k} c_i^2 \lambda_i \cdot (1) > 0$$

Because λ_i's are positive.

It suffices to show that for any $\vec{z} \neq \vec{0}$ in $U^\perp$, $\vec{z}^T A \vec{z} = 0$, which would mean we cannot share a nonzero vector with $\text{Eig}_+(A)$.

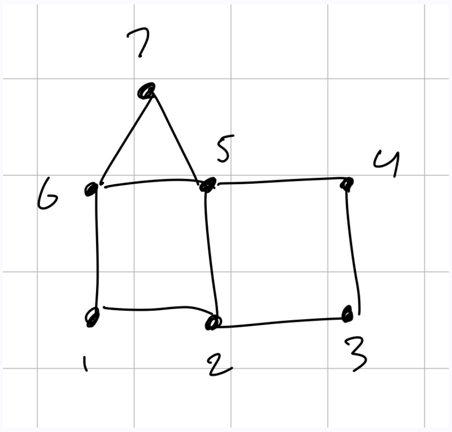
Recall that $U^\perp = \{\vec{x} \in \mathbb{R}^n : \vec{x}^T \vec{u}_i = \vec{x} \cdot \vec{u}_i = 0 \text{ for all } 1 \leq i \leq m\}$
Now for $\vec{z} \in U^\perp$,

$$\begin{aligned}
\vec{z}^T A \vec{z} &= \vec{z}^T \left(\sum_{i=1}^{m} A(B(X_i, Y_i)) \right) \vec{z} \\
&= \vec{z}^T \left(\sum_{i=1}^{m} (\vec{u}_i \vec{v}_i^T + \vec{v}_i \vec{u}_i^T) \right) \vec{z} \\
&= \sum_{i=1}^{m} (\vec{z}^T \vec{u}_i \vec{v}_i \vec{z} + \vec{z}^T \vec{v}_i \vec{u}_i^T \vec{z}) \\
&= \sum_{i=1}^{m} (0 \cdot \vec{v}_i \vec{z} + \vec{z}^T \vec{v}_i \cdot 0) \\
&= 0
\end{aligned}$$

$$\implies \text{Eig}_+(A) \cap U^\perp = \{\vec{0}\} \implies \text{Eig}_+(A) \subseteq U \implies \dim(\text{Eig}_+(A)) + \dim(U^\perp) \leq n$$
$$\implies \dim(\text{Eig}_+(A)) + \dim(U^\perp) - m \leq n - m \leq \dim(U^\perp)$$
$$\implies \# \text{ of bicliques} = m \geq \dim(\text{Eig}_+(A))$$

□

Example 37.1
Take the following graph:

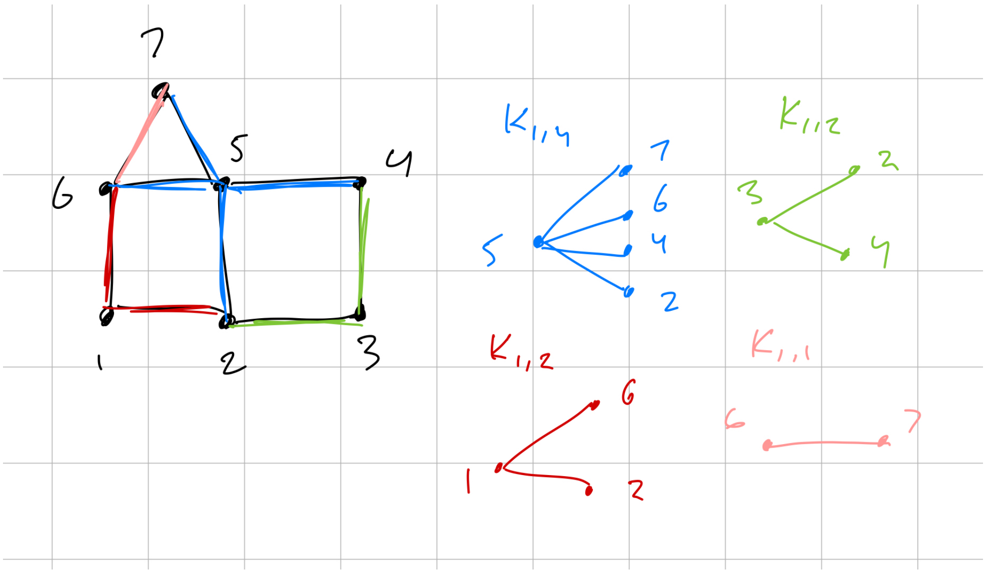


And the following spectrum candidates:

- $\{-3.1, (-2.44)^{(2)}, 0, (2.44)^2, 3.1\}$
- $\{-2.77, -0.1, (0.2)^{(3)}, 1.03, 1.24\}$
- $\{-2.42, -1.37, -0.6, -0.38, 0.76, 1.27, 2.74\}$

Which is the spectrum?

We can perform the following biclique decomposition:



Note that this gives us an upper bound on $\text{bp}(G)$, the minimum number of bicliques that a graph can be decomposed into. So, $\text{bp}(G) \leq 4$, so the spectrum can not be 2 because the theorem states that

$$\text{bp}(G) \geq \max(\{\# \text{ pos}, \# \text{ neg eigen values}) \geq \max(\{2, 5\}) = 5$$

Also, it cannot be the first spectrum since G is not bipartite because $5 \to 6 \to 7 \to 5$ is an odd cycle. So, the spectrum must be the third option.

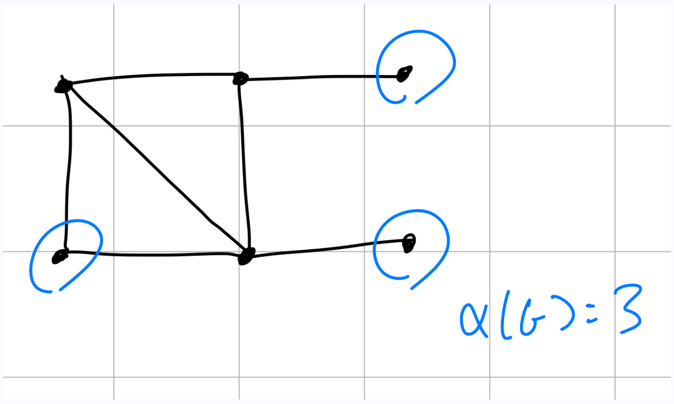
Moreover, by the spectrum, $\text{bp}(G) \geq \max(\{3, 4\}) = 4$, so combining this with the above construction, $\text{bp}(G) = 4$.

37.1 Independence (Section 5.5)

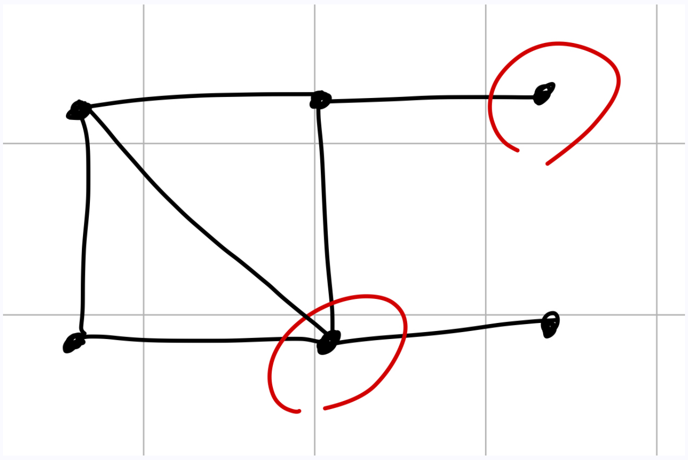
Definition 37.2
A subset S of vertices of a graph G is an **independent set** if for any $u, v \in S$, u is not adjacent to v.

The **independence number** of G, denoted $\alpha(G)$, is the size of the largest independent set.

Example 37.3
In the following graph, the circled vertices make up a maximum independent set.



The following vertices make up a maximal independent set, a set of vertices to which we can not any more if we wish to maintain the property of being an independent set.



Definition 37.4
Let $\vec{x}_S$ denote the characteristic vector of an independent set:

$$(\vec{x}_S)_i = \begin{cases} 1 & i \in S \\ 0 & \text{otherwise} \end{cases}$$

Lemma 37.5
For any independent set S,

$$|S| = \sum_{i=1}^{n} a_i^2$$

where

$$\vec{x}_S = \sum_{i=1}^{n} a_i \vec{v}_i$$

where $\{\vec{v}_1, \cdots, \vec{v}_n\}$ is an orthonormal basis of eigenvectors of $A(G)$.

Proof.

$$|S| = \vec{x}_S \cdot \vec{x}_S = \left(\sum_{i=1}^{n} a_i \vec{v}_i\right) \cdot \left(\sum_{i=1}^{n} a_i \vec{v}_i\right) = \sum_{i=1}^{n} a_i^2 (\vec{v}_i \cdot \vec{v}_i) + \sum_{i \neq j} a_i a_j (\vec{v}_i \cdot \vec{v}_j) = \sum_{i=1}^{n} a_i^2$$

□

Lemma 37.6
If G is d-regular, $\lambda_1 = d$, then if $\vec{x}_S = \sum a_i \vec{v}_i$, then $a_1 = \vec{v}_1 \cdot \vec{x}_S = \frac{|S|}{\sqrt{n}}$.

Theorem 37.7 (Hoffman Ratio Bound)
If G is a connected d-regular graph with eigenvalues $d = \lambda_1 > \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$, then

$$\alpha(G) \leq \frac{n}{1 - \frac{d}{\lambda_n}}$$

Where λ_n is the most negative eigenvalue.