36 Biclique Decomposition

From last time, we had a **biclique**, or complete bipartite graph $K_{a,b}$. We decomposes the edges of a graph G into bicliques.

Because $K_{1,1}$ is just a single edge, it is always possible to decompose the edges of a graph into bicliques. We wish to find the minimum number of these bicliques.

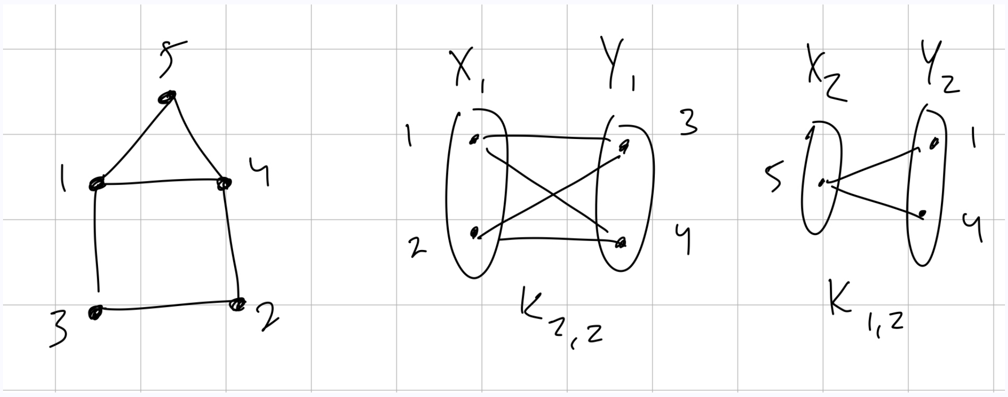
An example yields an upper bound, but how does one find a lower bound?

Definition 36.1
Let G be bipartite on n vertices with partite sets X and Y.

The **characteristic vector of** X is an n-dimensional column vector whose ith entry is 1 if $i \in X$, and 0 otherwise.

We denote this vector by $\vec{u}$. For Y, denote by v.

Example 36.2
Take the following graph G and its decomposition into bicliques:



In this example, $\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

$$
\begin{aligned}
\implies \vec{u}_1\vec{v}_1^T + \vec{v}_1\vec{u}_1^T &= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ & & 0 & & \\ & & 0 & & \\ & & 0 & & \end{bmatrix} + \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= A(B(X_1, Y_1)) = \text{adjacency matrix of the biclique}
\end{aligned}
$$

Lemma 36.3
If $B(X_i, Y_i)$ is a biclique with characteristic vectors $\vec{u}_i$ and $\vec{v}_i$, then the adjacency matrix is

$$A(B(X_i, Y_i)) = \vec{u}_i\vec{v}_i^T + \vec{v}_i\vec{u}_i^T$$

Note 36.4
In $\mathbb{R}^n$, $\dim(U) + \dim(U^\perp) = n$.

Theorem 36.5 (Witsenhausen)
Let $\text{bp}(G)$ denote the biclique number, the fewest (minimum) bicliques needed to decompose the edges of G.

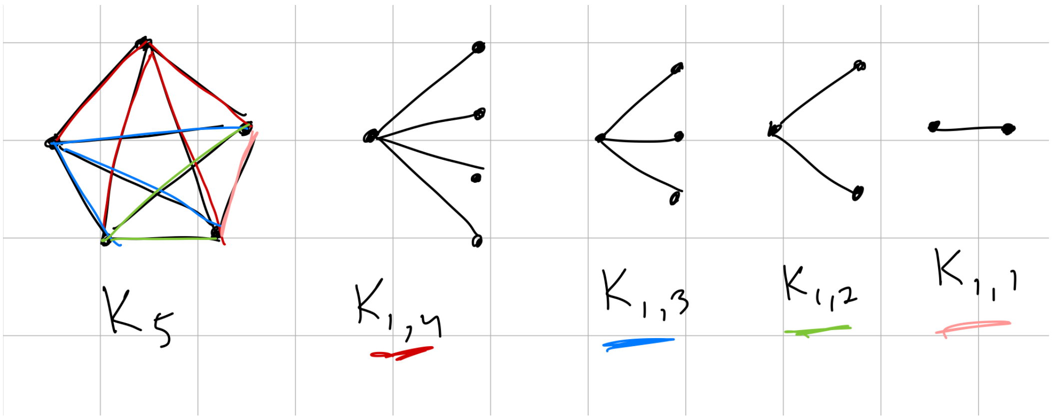
Let $\text{Eig}_+(A)$ denote the space spanned by the eigenvectors with (strictly) positive eigenvalues. Let $\text{Eig}_-(A)$ be defined similarly.

Then, by the spectral theorem,

$$\begin{aligned}\text{bp}(G) &\geq \max(\{\dim(\text{Eig}_+(A)), \dim(\text{Eig}_-(A))\}) \\ &= \max(\{\text{number of positive eigenvalues}, \text{number of negative eigenvalues}\})\end{aligned}$$

Example 36.6
What is $\text{bp}(K_n)$, where K_n is the complete graph with n vertices?

Note for K_5, we can decompose the edges into 4 bicliques with the following construction:

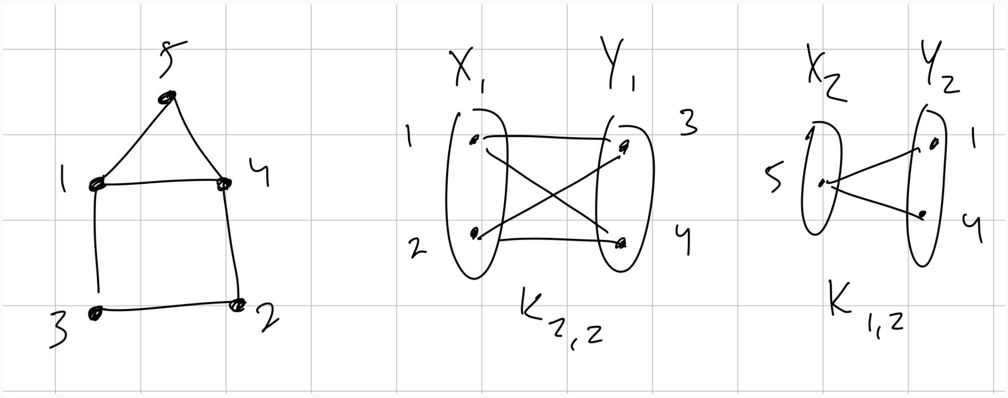


So we can see that $\text{bp}(K_5) \leq 4$.

This example construction shows we need at most $n-1$ bicliques, i.e., $\text{bp}(k_n) \leq n-1$.
Recall the spectrum of K_n is $\{n-1, (-1)^{(n-1)}\}$.
From the theorem above, we stated that $\text{bp}(G) \geq \max(\{\#\text{ positive}, \#\text{ negative eigenvalues}\}) = \max(\{1, n-1\})$.
So, $\text{bp}(K_n) \geq n-1$.

Combining with our earlier result for the upper bound, we have that $\text{bp}(K_n) = n-1$.

Example 36.7
Recall our earlier graph:



We showed that at most 2 bicliques are needed (1 is not enough since the graph is not a biclique).

The spectrum is $\{-2, -\sqrt{2}, (1)^{(2)}, \sqrt{2}\}$.

From our theorem, we get that $\text{bp}(G) \geq \max(\{2, 3\}) = 3$.

This is a contradiction, so this could not have been the spectrum.

Proof. Let $\text{bp}(G) = m$. We must show that $m \geq \max(\{\dim(\text{Eig}_+(A)), \dim(\text{Eig}_-(A))\})$.

We show it for $\text{Eig}_+(A)$.
By the spectral theorem, let $\{\vec{w}_1, \cdots, \vec{w}_k\}$ be an orthonormal basis of eigenvectors for $\text{Eig}_+(A)$. (We want to show $m \geq k = \dim(\text{Eig}_+(A))$)

Let $U = \text{span}(\{\vec{u}_1, \cdots, \vec{u}_m\})$ = span of characteristic vectors of the X_i's.
We don't know if these characteristic vectors are linearly independent, so $\dim(U) \leq m$.
Since $\dim(U) + \dim(U^\perp) = n$, $\dim(U_\perp) \geq n - m$.

Our main goal is to show $\text{Eig}_+(A) \cap U^\perp = \{\vec{0}\}$. If we show this, then $\text{Eig}_+(A) \subseteq U$, so $\dim(\text{Eig}_+(A)) + \dim(U^\perp) \leq n$.

$$\implies \dim(\text{Eig}_+(A)) + \dim(U^\perp) - m \leq n - m \leq \dim(U^\perp)$$
$$\implies \dim(\text{Eig}_+(A)) - m \leq 0$$
$$\implies \#\text{ of bicliques} = m \geq \dim(\text{Eig}_+(A))$$

as claimed. We will continue this next time. ☐