35 Bipartite Graphs (Section 5.4)

35.1 Bipartite Graphs (Section 5.4)

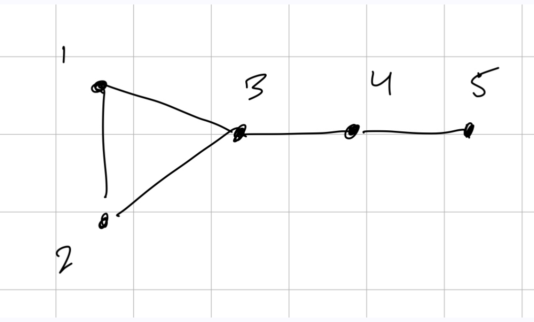
Definition 35.1
Graph G is **bipartite** if the vertex set can be partitioned into 2 disjoint subsets (partite sets) such taht every edge has one end point in each of the partite sets.

We can think of adjacent nodes as "alternating" between two sets A and B, and notice that if we have an odd cycle, then there is no way our graph can be bipartite. We can state this formally:

Theorem 35.2
Graph G is bipartite if and only if there are no odd cycles.

Theorem 35.3
Graph G is bipartite if and only if the spectrum is symmetric about 0, i.e, if λ is an eigenvalue, then $-\lambda$ is also an eigenvalue (including multiplicity).

Example 35.4
Take the following graph:



And the following candidates for the spectrum:

1. $\{-3.1, -1, 0, 2, 2.1\}$
2. $\{-2, -1.2, 0, 0.7, 2.5\}$
3. $\{-2.4, -1.7, 0, 1.7, 2.4\}$
4. $\{(-2.7)^{(2)}, 1, (2.2)^{(2)}\}$

Which one is the spectrum?

We know that (3) can not be the spectrum: the graph is bipartite (because there is an odd length cycle), so the spectrum must not be symmetric.

Recall that the eigenvalues of a graph are bounded by the highest degree $\Delta(G) = 3$ up to absolute value. So, (1) can not be the spectrum because it has the eigenvalue -3.1.

The diameter of the graph, the largest distance between any two nodes, is 3. So there must be at least 4 distinct eigenvalues in the graph, so the spectrum can not be (4).

So, the spectrum must be (2).

Proof. ($\implies$) Show that G being bipartite implies that λ and $-\lambda$ are eigenvalues.
Since G is bipartite, reorder the vertices so that the adjacency matrix is (with the left and upper sides corre-

sponding to a partite set A, and the right and lower sides corresponding to partite set B):

$$A(G) = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix}$$

Note that this is because nodes within the same partite set will not have any edges to each other, and an adjacency matrix is symmetric, which is why C and C^T are there.

Suppose λ is an eigenvalue of $A(G)$ with eigenvector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, where x and y are vectors of appropriate lengths.
For example, if partite sets A and B have sizes 2 and 3 respectively, and if $\vec{v} = (1, 2, 3, 4, 5)^T$, then $x = (1, 2)^T$ and $y = (3, 4, 5)^T$. Think of this as $\vec{v}$ being partitioned according to where the matrix is divided horizontally.

First, observe

$$A(G)\vec{v} = \lambda\vec{v} \implies \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{bmatrix} Cy \\ C^T x \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \implies Cy = \lambda x, C^T x = \lambda y$$

Let $\vec{v}_2 = \begin{bmatrix} x \\ -y \end{bmatrix}$. Then,

$$A(G)\vec{v}_2 = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix} \begin{bmatrix} x \\ -y \end{bmatrix} = \begin{bmatrix} -Cy \\ C^T x \end{bmatrix} = \begin{bmatrix} -\lambda x \\ \lambda y \end{bmatrix} = -\lambda \begin{bmatrix} x \\ -y \end{bmatrix} \implies A(G)\vec{v}_2 = -\lambda\vec{v}_2$$

So, $-\lambda$ is also an eigenvalue!

($\Longleftarrow$) Given that the spectrum is symmetric, we will show the graph is bipartite.

Le $\lambda_1, \cdots, \lambda_n$ be the eigenvalues (not necessarily distinct) with eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$. Note that

$$A\vec{v}_i = \lambda_i\vec{v}_i \implies A^2\vec{v}_i = A(A\vec{v}_i) = \lambda_i^2\vec{v}_i$$

So in general, $A^k\vec{v}_i = \lambda_i^k\vec{v}_i$.

Now for k odd,

$$\text{tr}(A^k) = \text{sum of eigenvalues of } A^k = \sum_{i=1}^{n} \lambda_i^k = 0$$

This is because the eigenvalues are symmetric about 0, so for odd powers, everything cancels.

Recall that trace is the sum of the diagonal entries, and A^k represents the number of walks of length k (odd). Then, the values in the diagonal must all be 0, which means there is no way to get back to a node in an odd number of steps, meaning there are no odd cycles, meaning the graph is bipartite. □

Stronger result: Let G be connected. Then if $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ are the eigenvalues, G is bipartite if and only if $\lambda_n = -\lambda_1$.

Definition 35.5
A **complete bipartite graph**, or **biclique**, denoted $K_{a,b}$, is a bipartite graph with partite sets A and B, $|A| = a$, $|B| = b$, such that every vertex in A is adjacent to every vertex in B.

We will look into decomposing the edges of G using bicliques.